UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS3120 Classical Mechanics and Electrodynamics
Day of exam: Tuesday, 7 June, 2016
Exam hours: 4 hours, beginning at 09:00
Permitted materials: Approved calculator
Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Formula Collection/Formelsamling FYS3120 (6 sider)
Language: This paper is available also in Norwegian

This examination paper consists of 3 problems on 4 pages
Make sure that your copy of the examination paper is complete before answering.

PROBLEM 1
A sliding pendulum
A long, thin bar is fixed at both ends, and is tilted at an angle α relative to the horizontal plane. A thin tube of mass m is threaded on the bar and can slide without friction along the bar. A small ball with the same mass m sits at the end of a thin rod. The rod, which we consider as massless, is at the other end fixed to the middle of the tube, as shown in the figure. The rod can oscillate freely about this point.

We consider all motion of the composite system to be restricted to the two dimensional vertical plane defined by the long bar. Use in the following the displacement s of the tube along the bar, and the angle θ of the pendulum rod relative to the vertical direction as generalized coordinates. The gravitational acceleration is g.

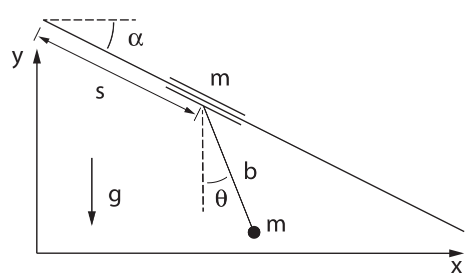

Figure 1: The sliding pendulum

a) Derive the Lagrangian of the system and show that it has the form

$$L = m\dot{s}^2 + mb\dot{s}\dot{\theta}\cos(\theta+\alpha) + \frac{1}{2}mb^2\dot{\theta}^2 + 2mgs\sin\alpha + mgb\cos\theta \tag{1}$$

b) Find Lagrange's equations for the two variables s and θ.

c) Show that s can be eliminated to give an equation for only θ and its time derivatives. Show that the equation has a solution where θ takes the constant value $\theta_0 = -\alpha$.

d) Find the solution for $\theta(t)$ under small oscillations about θ_0, and determine the corresponding angular frequency.

PROBLEM 2
Particle in circular orbit

An electron is circulating with constant speed in an accelerator ring. The radius of the ring is $R =$

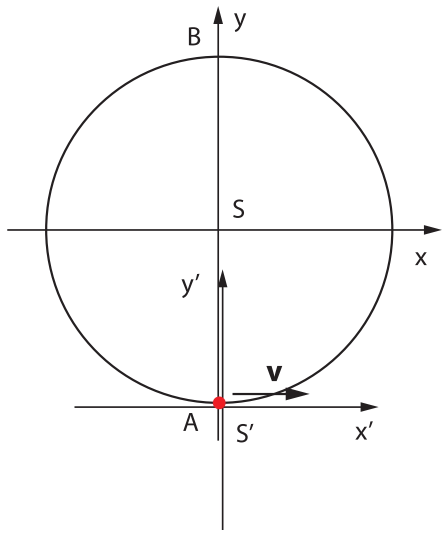

Figure 2: The orbit of the circulating electron.

$10\,m$, and the speed of the electron corresponds to a gamma factor $\gamma = 100$.

The laboratory frame S is the rest frame of the accelerator ring, and we assume that in the corresponding Cartesian coordinate frame the ring lies in the x, y-plane with the center of the ring at the origin. Since the electron is restricted to move in this plane we will in the following simply neglect the z coordinate and treat space-time as three-dimensional, with coordinates (ct, x, y). At time $t = 0$ the electron has coordinates $x = 0, y = -R$. This is shown as point A in the figure.

a) What is the velocity v of the electron in the lab frame S, measured relative to the speed of light c? Determine the circular frequency ω and the acceleration a of the electron in reference frame S. Find the coordinates of the electron's world line in S, $x^\mu(\tau)$, $\mu = 0, 1, 2$, expressed as functions of R, ω, γ and the *proper time* τ of the electron. Assume $\tau = 0$ at time $t = 0$.

b) Determine the components of the four-velocity $U^\mu(\tau)$ and the four-acceleration $A^\mu(\tau)$. Find the *proper acceleration* a_0 of the electron, and compare with the acceleration a measured in S.

We introduce another coordinate frame S', which is the instantaneous inertial rest frame of the electron at $t = 0$, when the electron is located in A. The corresponding space-time point is taken as origin of S'.

c) Explain what is meant by the *instantaneous inertial rest frame*, and give the transformation between the Cartesian coordinates of the two inertial frames S and S'. At time $t' = 0$ the accelerator ring defines a deformed circle in S'. Show, by use of the coordinate transformation, that it is an ellipse and determine the lengths of the long and short axes.

PROBLEM 3
Radiation from a linear antenna
The figure shows a linear antenna of length $2a$ lying along the z-axis with its center at the origin. We assume that the charge of the antenna is at all times located at the endpoints. The current in the antenna (between the charged end points) is given by $I = I_0 \sin \omega t$, where ω and I_0 are constants. The antenna is electrical neutral at time $t = 0$. The point A where the field is evaluated is given by the position vector $\mathbf{r}$, expressed in spherical coordinates as (r, θ, ϕ). The corresponding orthonormal vector basis is $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$. We will in the following assume that the point A is far away from the antenna, $r >> a$, and that the radiation from the antenna can be treated as electric dipole radiation.

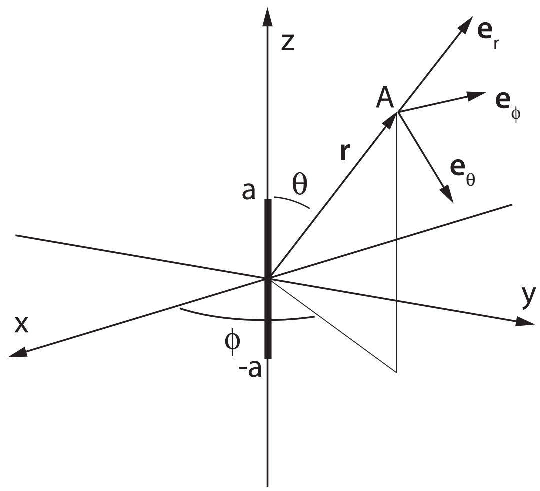

Figure 3: The linear antenna

a) Show that the antenna's electrical dipole moment at time t is given by

$$\mathbf{p}(t) = \frac{2aI_0}{\omega}(1 - \cos \omega t)\mathbf{k} \tag{2}$$

where $\mathbf{k}$ is the unit vector in the z-direction.

b) Find the radiation fields $\mathbf{B}$ and $\mathbf{E}$ at point A, expressed in the spherical vector basis, and written as functions of time t. What is the polarization of the radiation field?

c) Find the expression for Poynting's vector. Show that the time average of the total radiated power in all directions kan be written as $\bar{P} = \frac{1}{2}RI_0^2$ and determine R (the radiation resistance). What is the time average of the total power consumed by the antenna if it has an 'ordinary' resistance R_0 as well?

d) Find the radiation resistance R for an antenna of length $2a = 5\,\text{cm}$, which is conducting a current with frequency $f = 150\,\text{MHz}$. What is the time average of the total radiated power when $I_0 = 30\,\text{A}$?

For some useful expressions, see next page

The general form of the electric dipole radiation fields are,

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi rc}\ddot{\mathbf{p}}_{ret} \times \mathbf{n}\,, \quad \mathbf{E}(\mathbf{r},t) = c\mathbf{B}(\mathbf{r},t) \times \mathbf{n} \tag{3}$$

with $\mathbf{n}$ as the unit vector in the direction from the dipole to the point where the fields are measured. The spherical unit vectors are,

$$\begin{aligned}
\mathbf{e}_r &= \cos\phi\sin\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\theta\mathbf{k} \\
\mathbf{e}_\theta &= \cos\phi\cos\theta\mathbf{i} + \sin\phi\cos\theta\mathbf{j} - \sin\theta\mathbf{k} \\
\mathbf{e}_\phi &= -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}
\end{aligned} \tag{4}$$

The numerical value for the vacuum permeability is, $\mu_0 = 4\pi \times\, 10^{-7}\,\mathrm{N/A^2}$.