Lecture notes for FYS5190/FYS9190 – Supersymmetry

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Chapter 1

Groups and algebras

The goal of these lectures is to introduce the basics of low-energy models of supersymmetry (SUSY) using the Minimal Supersymmetric Standard Model (MSSM) as a main example. Rather than starting with the problems of the SM, we will focus on the algebraic origin of SUSY in the sense of an extension of the symmetries of Einsten's Special Relativity (SR), which was the original motivation for SUSY.

1.1   What is a group?

> **Definition**: The set $G = \{g_i\}$ and operation $\bullet$ form a **group** if and only if for $\forall\, g_i \in G$
>
> i) $g_i \bullet g_j \in G$ (closure)
>
> ii) $(g_i \bullet g_j) \bullet g_k = g_i \bullet (g_j \bullet g_k)$ (associativity)
>
> iii) $\exists e \in G$ such that $g_i \bullet e = e \bullet g_i = g_i$ (identity element)
>
> iv) $\exists g_i^{-1} \in G$ such that $g_i \bullet g_i^{-1} = g_i^{-1} \bullet g_i = e$ (inverse)

A simple example of a group is $G = \mathbb{Z}$ with usual addition as the operation, $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to $\mathbb{Z}_n$, where the operation is addition with modulo n. In this group, $g_i^{-1} = n - g_i$ and the unit element is $e = 0$. Note that $\mathbb{Z}$ is an *infinite* group, while $\mathbb{Z}_n$ is finite, with *order* n (meaning n members). Both are *abelian* groups, meaning that $g_i \bullet g_j = g_j \bullet g_i$.

All of this is "only" mathematics. Physicists are often more interested in groups where the elements of G *act* on some elements of a set $s \in S$, $g(s) = s' \in S$.[1] S here can for example be the state of a system, say a wave-function in quantum mechanics. We will return to this in a moment, let us just mention that the operation $g_i \bullet g_j$ acts as $(g_i \bullet g_j)(s) = g_i \bullet (g_j(s))$ and the identity acts as $e(s) = s$.[2]

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[1]As a result mathematics courses in group theory are not always so relevant to a physicist.
[2]We can prove this from iii) in the definition. Note that we use e as the identity in an abstract group, while

A more sophisticated example of a group can be found in a use for the Taylor expansion[3]

$$
\begin{aligned}
f(x+a) &= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \ldots \\
&= \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{dx^n} f(x) \\
&= e^{a\frac{d}{dx}} f(x)
\end{aligned}
$$

The operator $T_a = e^{a\frac{d}{dx}}$ is called the **translation operator** (in this case in one dimension). Together with the operation $T_a \bullet T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where $T_a^{-1} = T_{-a}$. In N dimensions the group $T(N)$ has the elements $T_{\vec{a}} = e^{\vec{a}\cdot\vec{\nabla}}$.

Definition: A subset $H \subset G$, is a **subgroup** if and only if:[a]

i) $h_i \bullet h_j \in H$ for $\forall h_i, h_j \in H$

ii) $h_i^{-1} \in H$ for $\forall h_i \in H$

---

[a]An alternative, more compact, way of writing these two requirements is $h_i \bullet h_j^{-1} \in H$ for $\forall h_i, h_j \in G$. This is often utilised in proofs.

Definition: H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$. A subgroup H is a **normal** (invariant) subgroup, if and only if for $\forall g \in G$,

$$
ghg^{-1} \in H \text{ for } \forall h \in H
$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no abelian normal subgroup.

The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U, i.e. matrices such that $U^\dagger U = 1$ or $U^{-1} = U^\dagger$. This has the neat property that for $\forall \vec{x}, \vec{y} \in \mathbb{C}^n$ multiplication by a unitary matrix leaves scalar products unchanged:

$$
\begin{aligned}
\vec{x}' \cdot \vec{y}' &\equiv \vec{x}'^\dagger \vec{y}' = (U\vec{x})^\dagger U\vec{y} \\
&= \vec{x}^\dagger U^\dagger U\vec{y} = \vec{x}^\dagger \vec{y} = \vec{x} \cdot \vec{y}
\end{aligned}
$$

If we additionally require that $\det(U) = 1$ the matrices form the **special unitary group** $SU(n)$. Let $U_i, U_j \in SU(n)$, then

$$
\det(U_i U_j^{-1}) = \det(U_i)\det(U_j^{-1}) = 1.
$$

---

1 is used as the identity matrix in matrix representations.

[3]This is the first of many points where any real mathematician would start to cry loudly and leave the room.

This means that $U_i U_j^{-1} \in SU(N)$. In other words, $SU(n)$ is a **proper subgroup** of $U(n)$. Let $V \in U(n)$ and $U \in SU(n)$, then $VUV^{-1} \in SU(n)$ because:

$$\det(VUV^{-1}) = \det(V)\det(U)\det(V^{-1}) = \frac{\det(V)}{\det(V)}\det(U) = 1.$$

In other words, $SU(n)$ is also a **normal subgroup** of $U(n)$.

> **Definition**: A **(left) coset** of a subgroup $H \subset G$ is a set $\{gh : h \in H\}$ where $g \in G$ and a **(right) coset** of a subgroup $H \subset G$ is a set $\{hg : h \in H\}$ where $g \in G$. For normal subgroups H the left and right cosets coincide and form the **coset group** G/H which has the members $\{gh : h \in H\}$ for $\forall g \in G$ and the binary operation $*$ with $gh * g'h' \in \{(g \bullet g')h : h \in H\}$.

> **Definition**: The **direct product** of groups G and H, $G \times H$, is defined as the *ordered pairs* (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet h_j)$. $G \times H$ is then a group and G and H are normal subgroups of $G \times H$.

> **Definition**: The **semi-direct product** $G \rtimes H$, where G is a mapping $G : H \to H$, is defined by the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet g_i(h_j))$. Here H is not a normal subgroup of $G \rtimes H$.

The SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is an example of a direct product. Direct products are "trivial" structures because there is no "interaction" between the subgroups. Can we imagine a group $G \supset SU(3)_c \times SU(2)_L \times U(1)_Y$ that can be broken down to the SM group but has a non-trivial unified gauge structure? There is, $SU(5)$ being one example.

1.2 Representations

> **Definition**: A **representation** of a group G on a vector space V is a map $\rho : G \to GL(V)$, where $GL(V)$ is the **general linear group** on V, i.e. invertible matrices of the field of V, such that for $\forall g_i, g_i \in G$, $\rho(g_i g_j) = \rho(g_i)\rho(g_j)$ (homeomorphism).

For $U(1)$ the transformation $e^{i\chi\alpha}$ is the **fundamental** or **defining representation** which can be used on wavefunctions $\psi(x)$—these form a one dimensional vector space over the complex numbers. For $SU(2)$ the transformation $e^{i\alpha_i \sigma_i}$, with σ being the Pauli matrices, is the **fundamental representation**, which can be applied to *e.g.* weak doublets $\psi = (\nu_l, l)$.[4]

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[4]This is a bit daft, since both $U(1)$ and $SU(2)$ are defined in terms of matrices. However, we will also have use for other representations, e.g. the **adjoint representation**, which is not the fundamental or defining representation.

Definition: Two representations ρ and ρ' of G on V and V' are **equivalent** if and only if $\exists A : V \to V'$, that is one-to-one, such that for $\forall g \in G$, $A\rho(g)A^{-1} = \rho'(g)$.

Definition: An **irreducible representation** ρ is a representation where there is *no* proper subspace $W \subset V$ that is closed under the group, i.e. there is no $W \subset V$ such that for $\forall w \in W$, $\forall g \in G$ we have $\rho(g)w \in W$.[a]

[a]In other words, we can not split the matrix representation of G in two parts that do not "mix".

Let $\rho(g)$ for $g \in G$ act on a vector space V as a matrix. If $\rho(g)$ can be decomposed into $\rho_1(g)$ and $\rho_2(g)$ such that

$$\rho(g)v = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix} v$$

for $\forall v \in V$, then ρ is **reducible**.

Definition: $T(R)$ is the **Dynkin index** of the representation R in terms of matrices T_a, given by $\text{Tr}[T_a, T_b] = T(R)\delta_{ab}$. $C(R)$ is the **Casimir invariant** given by $C(R)\delta_{ij} = (T^a T^a)_{ij}$

1.3   Lie groups

We begin by defining what we mean by Lie groups

Definition: A **Lie group** G is a finite-dimensional, n, **smooth manifold** C^∞, i.e. for $\forall g \in G$, g can locally be mapped onto (parametrised by) $\mathbb{R}^n$ or $\mathbb{C}^n$, and group multiplication and inversion are smooth functions, meaning that given $g(\vec{a}), g'(\vec{a}) \in G$, $g(\vec{a}') \bullet g'(\vec{a}') = g''(\vec{b})$ where $\vec{b}(\vec{a}, \vec{a}')$ is analytic, and $g^{-1}(\vec{a}) = g'(\vec{a}')$ where $\vec{a}'(\vec{a})$ is analytic.

In terms of a Lie group G acting on a vector space V, $\dim(V) = m$ (or more generally an m-dimensional manifold), this means we can write the map $G \times V \to V$ for $\vec{x} \in V$ as $x_i \to x_i' = f_i(x_i, a_j)$ where f_i is analytic in x_i and a_j. Additionally f_i should have an inverse.

The translation group $T(1)$ with $g(a) = e^{a\frac{d}{dx}}$ is a Lie group since $g(a) \cdot g(a') = g(a + a')$ and $a + a'$ is analytic. Here we can write $f(x, a) = x + a$. $SU(n)$ are Lie groups as they have a fundamental representation $e^{i\vec{\alpha}\vec{\lambda}}$ where λ is a set of $n \times n$-matrices, and $f_i(\vec{x}, \vec{\alpha}) = [e^{i\vec{\alpha}\vec{\lambda}}\vec{x}]_i$.

By the analyticity we can always construct the parametrization so that $g(0) = e$ or $x_i =$

$f_i(x_i, 0)$. By an infinitesimal transformation da_i we then get the following Taylor expansion[5]

$$
\begin{aligned}
x_i' &= x_i + dx_i = f_i(x_i, da_i) \\
&= f_i(x_i, 0) + \frac{\partial f_i}{\partial a_j} da_j + \ldots \\
&= x_i + \frac{\partial f_i}{\partial a_j} da_j
\end{aligned}
$$

This is the transformation by the member of the group that in the parameterisation sits da_j from the identity. If we now let F be a function from the vector space V to either the real $\mathbb{R}$ or complex numbers $\mathbb{C}$, then the group transformation defined by da_i changes F by

$$
\begin{aligned}
dF &= \frac{\partial F}{\partial x_i} dx_i \\
&= \frac{\partial F}{\partial x_i} \frac{\partial f_i}{\partial a_j} da_j \\
&\equiv da_j X_j F
\end{aligned}
$$

where the operators defined by

$$
X_j \equiv \frac{\partial f_i}{\partial a_j} \frac{\partial}{\partial x_i}
$$

are called the n **generators** of the Lie group. It is these generators X that define the action of the Lie group in a given representation as the a's are mere parameters.

As an example of the above we can now go in the opposite direction and look at the two-parameter transformation *defined* by

$$
x' = f(x) = a_1 x + a_2,
$$

which gives

$$
X_1 = \frac{\partial f}{\partial a_1} \frac{\partial}{\partial x} = x \frac{\partial}{\partial x},
$$

which is the generator for **dilation** (scale change), and

$$
X_2 = \frac{\partial}{\partial x},
$$

which is the generator for $T(1)$. Note that $[X_1, X_2] = -X_2$.

> **Exercise**: Find the generators of $SU(2)$ and their commutation relationships.
> Hint: One answer uses the Pauli matrices, but try to derive this from an infinitesimal parametrization.

Next we lists three central results on Lie groups derived by Sophus Lie [1]:

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[5] The fact that f_i is analytic means that this Taylor expansion must converge in some radius around $f_i(x_i, 0)$.

Theorem: (Lie's theorems)

 i) For a Lie group $\frac{\partial f_i}{\partial a_j}$ is analytic.

 ii) The generators X_i satisfy $[X_i, X_j] = C_{ij}^k X_k$, where C_{ij}^k are **structure constants**.

 iii) $C_{ij}^k = -C_{ji}^k$ and $C_{ij}^k C_{kl}^m + C_{jl}^k C_{ki}^m + C_{li}^k C_{kj}^m = 0.$[a]

---

[a]The second identity follows from the Jacobi identity $[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0$

Exercise: What are the structure constants of SU(2)?

1.4   Lie algebras

Definition: An **algebra** A on a field (say $\mathbb{R}$ or $\mathbb{C}$) is a linear vector space with a binary operation $\circ : A \times A \to A$.

The vector space $\mathbb{R}^3$ together with the cross-product constitutes an algebra.

Definition: A **Lie algebra** L is an algebra where the binary operator $[\,,\,]$, called Lie bracket, has the properties that for $x, y, z \in L$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$):

 i) (associativity)
$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y]$$

 ii) (anti-commutation)
$$[x, y] = -[y, x]$$

 iii) (Jacobi identity)
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

We usually restrict ourselves to algebras of linear operators with $[x, y] = xy - yx$, where property iii) is automatic. From Lie's theorems the generators of an n-dimensional Lie group

form an n-dimensional Lie algebra.

We mentioned the fundamental representation of a matrix based group earlier. These representations have the lowest possible dimension. Another important representation is the **adjoint**. This consists of the matrices:

$$(M_i)^k_j = -C^k_{ij}$$

where C^k_{ij} are the structure constants. From the Jacobi identity we have $[M_i, M_j] = C^k_{ij} M_k$, meaning that the adjoint representation fulfills the same algebra as the fundamental (generators). Note that the dimension of the fundamental representation n for $SO(n)$ and $SU(n)$ is always smaller than the adjoint, which is equal to the degrees of freedom, $\frac{1}{2}n(n-1)$ and $n^2 - 1$ respectively.

Exercise: Find the dimensions of the fundamental and adjoint representations of $SU(n)$.

Exercise: Find the fundamental representation for $SO(3)$ and the adjoint representation for $SU(2)$. What does this say about the groups and their algebras?

Chapter 2

The Poincaré algebra and its extensions

We now take a look at the groups behind Special Relativity (SR), the Lorentz and Poincaré groups, and look for ways to extend them to internal symmetries, *i.e.* gauge groups.

2.1 The Lorentz Group

A point in the Minkowski space-time manifold $\mathbb{M}_4$ is given by $x^\mu = (t, x, y, z)$ and Einstein's requirement was that physics should be invariant under the Lorentz group.

> **Definition**: The **Lorentz group** L is the group of linear transformations $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that $x^2 = x_\mu x^\mu = x'_\mu x'^\mu$ is invariant. The **proper orthochronous Lorentz group** $L_+^\uparrow$ is a subgroup of L where $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$.
> a
>
> ────────────────────
>
> ^{a}This guarantees that time moves forward, and makes space and time reflections impossible, with the group describing only boosts and rotations.

From the discussion in the previous section one can show that any $\Lambda \in L_+^\uparrow$ can be written as

$$\Lambda^\mu{}_\nu = \left[\exp\left(-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)\right]^\mu{}_\nu, \tag{2.1}$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ are the parameters of the transformation and $M_{\rho\sigma}$ are the generators of L, and the basis of the Lie algebra for L, and are given by:

$$M = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix},$$

where K_i and J_i are generators of boost and rotation respectively. These fulfil the following algebra:[1]

$$[J_i, J_j] = -i\epsilon_{ijk}J_k, \tag{2.2}$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k, \tag{2.3}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.4}$$

The generators M of L obey the commutation relation:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \tag{2.5}$$

2.2   The Poincaré group

We extend L by translation to get the Poincaré group, where translation : $x^\mu \to x'^\mu = x^\mu + a^\mu$. This leaves lengths $(x - y)^2$ invariant in $\mathbb{M}_4$.

> **Definition**: The **Poincaré group** P is the group of all transformations of the form
>
> $$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$
>
> We can also construct the **restricted Poincaré group** $P_+^\uparrow$, by restricting the matrices Λ in the same way as in $L_+^\uparrow$.

We see that the composition of two elements in the group is:

$$(\Lambda_1, a_1) \bullet (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1).$$

This tells us that the Poincaré group is **not** a direct product of the Lorentz group and the translation group, but a **semi-direct product** of L and the translation group $T(1,3)$, $P = L \rtimes T(1,3)$. The translation generators P_μ have a trivial commutation relationship:[2]

$$[P_\mu, P_\nu] = 0 \tag{2.6}$$

One can show that:[3]

$$[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \tag{2.7}$$

Equations (2.5)–(2.7) form the **Poincaré algebra**, a Lie algebra.

2.3   The Casimir operators of the Poincaré group

> **Definition**: The **Casimir operators** of a Lie algebra are the operators that commute with all elements of the algebra [a]
>
> ---
>
> [a]Technically we say they are members of the centre of the universal enveloping algebra of the Lie algebra. Whatever that means.

---

[1]Notice that (2.2) and (2.4) are the $SU(2)$ algebra.

[2]This means that the translation group in Minkowski space is abelian. This is obvious, since $x^\mu + y^\mu = y^\mu + x^\mu$. One can show that the differential representation is the expected $P_\mu = -i\partial_\mu$.

[3]For a rigorous derivation of this see Chapter 1.2 of [2]

A central theorem in representation theory for groups and algebras is **Schur's lemma**:

Theorem: (Schur's Lemma)
In any irreducible representation of a Lie algebra, the Casimir operators are proportional to the identity.

This has the wonderful consequence that the constants of proportionality can be used to classify the (irreducible) representations of the Lie algebra (and group). Let us take a concrete example to illustrate: $P^2 = P_\mu P^\mu$ is a Casimir operator of the Poincaré algebra because the following holds:[4]

$$\left[P_\mu, P^2\right] = 0, \tag{2.11}$$
$$\left[M_{\mu\nu}, P^2\right] = 0. \tag{2.12}$$

This allows us to label the irreducible representation of the Poincaré group with a quantum number m^2, writing a corresponding state as $|m\rangle$, such that:[5]

$$P^2|m\rangle = m^2|m\rangle.$$

The number of Casimir operators is the **rank** of the algebra, *e.g.* rank $SU(n) = n - 1$. It turns out that $P_+^\uparrow$ has rank 2, and thus two Casimir operators. To demonstrate this is rather involved, and we won't make an attempt here, but note that it can be shown that[6] $L_+^\uparrow \cong SU(2) \times SU(2)$ because of the structure of the boost and rotation generators, where $SU(2)$ can be shown to have rank 1. Furthermore, $L_+^\uparrow \cong SL(2, \mathbb{C})$. We will return to this relationship between $L_+^\uparrow$ and $SL(2, \mathbb{C})$ in Section 2.5, where we use it to reformulate the algebras we work with in supersymmetry.

So, what is the second Casimir of the Poincaré algebra?

Definition: The **Pauli-Ljubanski polarisation vector** is given by:

$$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{2.13}$$

---

[4]The first relation follows trivially from the commutation of P_μ with P_ν. To show the second we first use that

$$[M_{\mu\nu}, P_\rho P^\rho] = [M_{\mu\nu}, P_\rho]P^\rho + P_\rho[M_{\mu\nu}, P^\rho], \tag{2.8}$$

and Eq. (2.7) to get:

$$[M_{\mu\nu}, P_\rho P^\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu)P^\rho - iP_\rho(g_\mu{}^\rho P_\nu - g_\nu{}^\rho P_\mu), \tag{2.9}$$

thus

$$[M_{\mu\nu}, P_\rho P^\rho] = -2i[P_\mu, P_\nu] = 0. \tag{2.10}$$

[5]This quantum number looks astonishingly like mass and P^2 like the square of the 4-momentum operator. However, we note that in general m^2 is not restricted to be larger than zero.
[6]Here $\cong$ means homomorfic, that is structure preserving.

Then $W^2 = W_\mu W^\mu$ is a Casimir operator of $P_+^\uparrow$, $i.e.$:

$$\left[M_{\mu\nu}, W^2\right] = 0 \qquad (2.14)$$
$$\left[P_\mu, W^2\right] = 0 \qquad (2.15)$$

To show this we can re-write the operator as:[7]

$$W^2 = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} P^2 + M^{\rho\sigma} M_{\nu\sigma} P_\rho P^\nu.$$

From the above it is easy to show that W^2 is indeed a Casimir

Again, because W^2 is a Casimir operator, we can label all states in an irreducible representation (read particles) with quantum numbers m, s, such that:

$$W^2|m, s\rangle = -m^2 s(s+1)|m, s\rangle$$

The m^2 appears because there are two P_μ operators in each term. However, what is the significance of the s, and why do we choose to write the quantum number in that (familiar?) way? One can easily show using ladder operators that $s = 0, \frac{1}{2}, 1, \ldots$, $i.e.$ can only take integer and half integer values. In the rest frame (RF) of the particle we have:[8]

$$P_\mu = (m, \vec{0})$$

Using that $WP = 0$ this gives us $W_0 = 0$ in the RF, and furthermore:

$$W_i = \frac{1}{2}\epsilon_{i0jk} m M^{jk} = mS_i,$$

where $S_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ is the **spin operator**. This gives $W^2 = -\vec{W}^2 = -m^2\vec{S}^2$, meaning that s is indeed the spin quantum number.[9]

The conclusion of this subsection is that anything transforming under the Poincaré group, meaning the objects considered by SR, can be classified by two quantum numbers: mass and spin.

2.4　The no-go theorem and graded Lie algebras

Since we now know the Poincaré group and its representations well, we can ask: Can the external space-time symmetries be extended, perhaps also to include the internal gauge symmetries? Unfortunately no. In 1967 Coleman and Mandula [3] showed that any extension of the Pointcaré group to include gauge symmetries is isomorphic to $G_{SM} \times P_+^\uparrow$, $i.e.$ the generators B_i of standard model gauge groups all have

$$[P_\mu, B_i] = [M_{\mu\nu}, B_i] = 0.$$

Not to be defeated by a simple mathematical proof this was countered by Haag, Łopuszański and Sohnius (HLS) in 1975 in [4] where they introduced the concept of graded Lie algebras

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[7]This is non-trivial to demonstrate, see Chapter 1.2 of [2].

[8]This does not loose generality since physics should be independent of frame.

[9]Observe that this discussion is problematic for massless particles. However, it is possible to find a similar relation for massless particles, when we chose a frame where the velocity of the particle is mono-directional.

to get around the no-go theorem.

Definition: A ($\mathbb{Z}_2$) **graded Lie algebra** or **superalgebra** is a vector space L that is a direct sum of two vector spaces L_0 and L_1, $L = L_0 \oplus L_1$ with a binary operation $\bullet : L \times L \to L$ such that for $\forall x_i \in L_i$

i) $x_i \bullet x_j \in L_{i+j \mod 2}$ (grading)[a]

ii) $x_i \bullet x_j = -(-1)^{ij} x_j \bullet x_i$ (supersymmetrization)

iii) $x_i \bullet (x_j \bullet x_k)(-1)^{ik} + x_j \bullet (x_k \bullet x_i)(-1)^{ji} + x_k \bullet (x_i \bullet x_j)(-1)^{kj} = 0$ (generalised Jacobi identity)

This definition can be generalised to $\mathbb{Z}_n$ by a direct sum over n vector spaces L_i, $L = \oplus_{i=0}^{n-1} L_i$, such that $x_i \bullet x_j \in L_{i+j \mod n}$ with the same requirements for supersymmetrization and Jacobi identity as for the $\mathbb{Z}_2$ graded algebra.

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[a]This means that $x_0 \bullet x_0 \in L_0$, $x_1 \bullet x_1 \in L_0$ and $x_0 \bullet x_1 \in L_1$.

We can start, as HLS, with a Lie algebra ($L_0 = P_+^\uparrow$) and add a new vector space L_1 spanned by four operators, the Majorana spinor charges Q_a. It can be shown that the superalgebra requirements are fulfilled by:

$$[Q_a, P_\mu] = 0 \tag{2.16}$$

$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu} Q)_a \tag{2.17}$$

$$\{Q_a, \bar{Q}_b\} = 2\not{P}_{ab} \tag{2.18}$$

where $\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and as usual $\bar{Q}_a = (Q^\dagger \gamma_0)_a$.[10]

Unfortunately, the internal gauge groups are nowhere to be seen. They can appear if we extend the algebra with Q_a^α, where $\alpha = 1, \ldots, N$, which gives gives rise to so-called $N > 1$ supersymmetries. This introduces extra particles and does not seem to be realised in nature due to an extensive number of extra particles.[11] This extension, including $N > 1$, can be proven, under some reasonable assumptions, to be the **largest possible** extension of SR.

2.5 Weyl spinors

Previously we claimed that there is a homomorphism between $L_+^\uparrow$ and $SL(2, \mathbb{C})$. This homomorphism, with $\Lambda^\mu{}_\nu \in L_+^\uparrow$ and $M \in SL(2, \mathbb{C})$, can be explicitly given by:[12]

$$\Lambda^\mu{}_\nu(M) = \frac{1}{2}\text{Tr}[\bar{\sigma}^\mu M \sigma_\nu M^\dagger], \tag{2.19}$$

$$M(\Lambda^\mu{}_\nu) = \pm\frac{1}{\sqrt{\det(\Lambda^\mu{}_\nu \sigma_\mu \bar{\sigma}^\nu)}}\Lambda^\mu{}_\nu \sigma_\mu \bar{\sigma}^\nu, \tag{2.20}$$

where $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and $\sigma^\mu = (1, \vec{\sigma})$.

---

[10]Alternatively, (2.18) can be written as $\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu$.

[11]Note that $N > 8$ would include particles with spin greater than 2.

[12]The sign in Eq. (2.20) is the reason that this is a homomorphism, instead of an isomorphism. Each element in $SL(2, \mathbb{C})$ can be assigned to two in $L_+^\uparrow$.

Since we have this homomorphism we can look at the representations of $SL(2,\mathbb{C})$ instead of the Poincaré group (with its usual Dirac spinors) when we describe particles, but what are those representations? It turns out that there exist two inequivalent fundamental representations of $SL(2,\mathbb{C})$:

i) The self-representation $\rho(M) = M$ working on an element ψ of a representation space F:

$$\psi'_A = M_A{}^B \psi_B \quad A, B = 1, 2$$

ii) The complex conjugate self-representation $\rho(M) = M^*$ working on $\bar{\psi}$ in a space $\dot{F}$:[13]

$$\bar{\psi}'_{\dot{A}} = (M^*)_{\dot{A}}{}^{\dot{B}} \bar{\psi}_{\dot{B}} \quad \dot{A}, \dot{B} = 1, 2$$

> **Definition**: ψ and $\bar{\psi}$ are called **left- and right-handed Weyl spinors**.

Indices can be lowered and raised with:

$$\epsilon_{AB} = \epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\epsilon^{AB} = \epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The relationship between ψ and $\bar{\psi}$ can be expressed with:[14]

$$\bar{\sigma}^{0\,\dot{A}A} (\psi_A)^* = \bar{\psi}^{\dot{A}}$$

Note that from the above:

$$(\psi_A)^\dagger = \bar{\psi}_{\dot{A}}$$
$$(\bar{\psi}_{\dot{A}})^\dagger = \psi_A$$

We define contractions of Weyl spinors as follows:

> **Definition**: $\psi\chi \equiv \psi^A \chi_A$ and $\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{A}} \bar{\chi}^{\dot{A}}$.

These quantities are invariant under $SL(2,\mathbb{C})$. With this in hand we see that

$$\psi^2 \equiv \psi\psi = \psi^A \psi_A = \epsilon^{AB} \psi_B \psi_A = \epsilon^{12} \psi_2 \psi_1 + \epsilon^{21} \psi_1 \psi_2 = \psi_2 \psi_1 - \psi_1 \psi_2.$$

This quantity is zero if the Weyl spinors commute. In order to avoid this we make the following assumption which is consistent with how we treat fermions (and Dirac spinors):

> **Postulate**: All Weyl spinors anticommute:[a] $\{\psi_A, \psi_B\} = \{\bar{\psi}_{\dot{A}}, \bar{\psi}_{\dot{B}}\} = \{\psi_A, \bar{\psi}_{\dot{B}}\} = \{\bar{\psi}_{\dot{A}}, \psi_B\} = 0$.
>
> _____
>
> [a]This means that Weyl spinors are so-called **Grassmann numbers**.

_____

[13]The dot on the indices is just there to help us remember which sum is which and does not carry any additional importance.

[14]This is a bit daft, as $\bar{\sigma}^{0\,\dot{A}A} = \delta_{\dot{A}A}$, and we will in the following omit the matrix and write $(\psi_A)^* = \bar{\psi}^{\dot{A}}$.

This means that

$$\psi^2 \equiv \psi\psi = \psi^A\psi_A = -2\psi_1\psi_2.$$

Weyl spinors can be related to Dirac spinors ψ_a as well:[15]

$$\psi_a = \begin{pmatrix} \psi_A \\ \bar{\chi}^{\dot{A}} \end{pmatrix}.$$

We see that in order to describe a Dirac spinor we need both handedness of Weyl spinor. For Majorana spinors we have:

$$\psi_a = \begin{pmatrix} \psi_A \\ \bar{\psi}^{\dot{A}} \end{pmatrix}.$$

We can now write the super-Poincaré algebra (superalgebra) in terms of Weyl spinors. With

$$Q_a = \begin{pmatrix} Q_A \\ \bar{Q}^{\dot{A}} \end{pmatrix}, \tag{2.21}$$

for the Majorana spinor charges, we have

$$\begin{aligned}
\{Q_A, Q_B\} &= \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 0 & \text{(2.22)} \\
\{Q_A, \bar{Q}_{\dot{B}}\} &= 2\sigma^{\mu}_{A\dot{B}} P_{\mu} & \text{(2.23)} \\
[Q_A, P_{\mu}] &= [\bar{Q}_{\dot{A}}, P_{\mu}] = 0 & \text{(2.24)} \\
[Q_A, M^{\mu\nu}] &= \sigma^{\mu\nu\,B}_A Q_B & \text{(2.25)}
\end{aligned}$$

where $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$.

Exercise: Show that $L^{\uparrow}_{+}$ and $SL(2, \mathbb{C})$ are indeed **homomorphic**, *i.e.* that the mapping defined by (2.19) or (2.20) has the property that $\Lambda(M_1 M_2) = \Lambda(M_1)\Lambda(M_2)$ or $M(\Lambda_1\Lambda_2) = M(\Lambda_1)M(\Lambda_2)$.

2.6 The Casimir operators of the super-Poincaré algebra

When Q_a is four-dimensional it is easy to see that P^2 is still a Casimir operator of the superalgebra. From Eq. (2.24) P_μ commutes with the Qs, so in turn P^2 must commute. However, W^2 is not a Casimir because of the following result:

$$[W^2, Q_a] = W_\mu(\not{P}\gamma_\mu\gamma^5 Q)_a + \frac{3}{4}P^2 Q_a.$$

We want to find an extension of W that commutes with the Qs while retaining the commutators we alread have. The construction

$$C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu,$$

---

[15]Note that in general $(\psi_A)^* \neq \bar{\chi}^{\dot{A}}$.

where

$$B_\mu \equiv W_\mu + \frac{1}{4}X_\mu,$$

with

$$X_\mu \equiv \frac{1}{2}\bar{Q}\gamma_\mu\gamma^5 Q,$$

has the required relation:

$$[C_{\mu\nu}, Q_a] = 0.$$

By excessive algebra we can show that:

$$\begin{aligned}
[C^2, Q_a] &= 0 \quad \text{(trivial)} \\
[C^2, P_\mu] &= 0 \quad \text{(algebra)} \\
[C^2, M_{\mu\nu}] &= 0 \quad \text{(because } C^2 \text{ is a Lorentz scalar)}
\end{aligned}$$

Thus C^2 is a Casimir operator for the superalgebra.

2.7   Representations of the superalgebra

What sort of particles are described by the superalgebra? Let us again assume without loss of generality that we are in the rest frame, *i.e.* $P_\mu = (m, \vec{0})$. As for the original Poincaré group, states are labeled by m, where m^2 is the eigenvalue of P^2. For C^2 we have to do a bit of calculation:

$$\begin{aligned}
C^2 &= 2B_\mu P_\nu B^\mu P^\nu - 2B_\mu P_\nu B^\nu P^\mu \\
&\overset{RF}{=} 2m^2 B_\mu B^\mu - 2m^2 B_0^2 \\
&= 2m^2 B_k B^k,
\end{aligned}$$

and from the definition of B_μ we get:

$$\begin{aligned}
B_k &= W_k + \frac{1}{4}X_k \\
&= mS_k + \frac{1}{8}\bar{Q}\gamma_\mu\gamma^5 Q \equiv mJ_k.
\end{aligned}$$

The operator we just defined, $J_k \equiv \frac{1}{m}B_k$, is an abstraction of the ordinary spin operator, and fulfills the angular momentum algebra (just like the spin operator):

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$

and has $[J_k, Q_a] = 0.$[16] This gives us

$$C^2 = 2m^4 J_k J^k,$$

such that:

$$C^2|m, j, j_3\rangle = -m^4 j(j+1)|m, j, j_3\rangle,$$

---

[16] Again the proof is algebraically extensive, and again I suggest the interested reader to pursue [2].

where it can be shown that $j = 0, \frac{1}{2}, 1 \ldots$ and $j_3 = -j, -j + 1, \ldots, j$ because J_k fulfils the angular momentum algebra. So, the irreducible representations of the superalgebra can be labeled by m, j, and any given set m, j will give us $2j + 1$ states with different j_3.[17]

In the following we will construct all the states for a given representation with the set m, j. To do this it is very usefull to write the generators Q in terms of two-component Weyl spinors instead of four-component Dirac spinors, making explicit use of their Majorana nature, as we did in Section 2.5. We note that from the above discussion

$$[J_k, Q_A] = [J_k, \bar{Q}_{\dot{B}}] = 0.$$

We begin by claiming that for any given j_3 there must then exist a state $|\Omega\rangle$ that has the same value of j_3 and for which

$$Q_A|\Omega\rangle = 0. \tag{2.26}$$

This is called the **Clifford vacuum**.[18] To show this, start with $|\beta\rangle$, a state with j_3. Then the construction

$$|\Omega\rangle = Q_1 Q_2 |\beta\rangle$$

has these properties. First we show that (2.26) holds:

$$Q_1 Q_1 Q_2 |\beta\rangle = -Q_1 Q_1 Q_2 |\beta\rangle = 0$$

and

$$Q_2 Q_1 Q_2 |\beta\rangle = -Q_1 Q_2 Q_2 |\beta\rangle = Q_1 Q_2 Q_2 |\beta\rangle = -Q_2 Q_1 Q_2 |\beta\rangle = 0.$$

For this Clifford vacuum state we then have:

$$\begin{aligned} J_3|\Omega\rangle &= J_3 Q_1 Q_2 |\beta\rangle \\ &= Q_1 Q_2 J_3 |\beta\rangle = j_3 |\Omega\rangle, \end{aligned}$$

in other words, $|\Omega\rangle$ has the same value for j_3 as the $|\beta\rangle$ it was constructed from. We can now use the explicit expression for J_k

$$J_k = S_k - \frac{1}{4m} \bar{Q}_{\dot{B}} \bar{\sigma}_k^{\dot{B}A} Q_A,$$

in order to find the spin for this state:

$$J_k|\Omega\rangle = S_k|\Omega\rangle = j_k|\Omega\rangle,$$

meaning that $s_3 = j_3$ and $s = j$ are the eigenvalues of S_3 and S^2 for the Clifford vacuum $|\Omega\rangle$.

We can construct three more states from the Clifford vacuum:[19]

$$\bar{Q}^{\dot{1}}|\Omega\rangle, \quad \bar{Q}^{\dot{2}}|\Omega\rangle, \quad \bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle.$$

This means that there are four possible states that can be constructed out of any state with the quantum numbers m, j, j_3. Taking a look at:

$$J_k \bar{Q}^{\dot{A}}|\Omega\rangle = \bar{Q}^{\dot{A}} J_k |\Omega\rangle = j_k \bar{Q}^{\dot{A}}|\Omega\rangle,$$

---

[17]Note that j is NOT the spin, but a generalization of spin.

[18]It is called the Clifford vacuum because the operators satisfy a Clifford algebra $\{Q_A, \bar{Q}_{\dot{B}}\} = 2m\sigma^0_{A\dot{B}}$. Do not confuse this with a vacuum state, it is only a name.

[19]All other possible combinations of Qs and $|\Omega\rangle$ give either one of the other four states, or the zero state which is trivial and of no interest.

this means that all these states have the same j_3 (and j) quantum numbers.[20]   From the superalgebra (2.25) we have:

$$[M^{ij}, \bar{Q}^{\dot{A}}] = -(\sigma^{ij})^{\dot{A}}{}_{\dot{B}}\bar{Q}^{\dot{B}},$$

so that:

$$S_3\bar{Q}^{\dot{A}}|\Omega\rangle = \bar{Q}^{\dot{A}}S_3|\Omega\rangle - \frac{1}{2}(\bar{\sigma}_3\sigma^0)^{\dot{A}}{}_{\dot{B}}\bar{Q}^{\dot{B}}|\Omega\rangle = \left(j_3 \mp \frac{1}{2}\right)\bar{Q}^{\dot{A}}|\Omega\rangle,$$

where $-$ is for $\dot{A} = \dot{1}$ and $+$ is for $\dot{A} = \dot{2}$. We can similarly show that

$$S_3\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle = j_3\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle.$$

This means that each set of quantum numbers m, j, j_3 gives 2 states with $s_3 = j_3$, and two with $s_3 = j_3 \pm \frac{1}{2}$, giving two bosonic and two fermionic states, with the same mass.

The above explains the much repeated statement that any supersymmetry theory has an equal number of bosons and fermions, which, incidentally, is not true.

Theorem: For any representation of the superalgebra where P_μ is a one-to-one operator there is an equal number of boson and fermion states.

To show this, divide the representation into two sets of states, one with bosons and one with fermions. Let $\{Q_A, \bar{Q}_{\dot{B}}\}$ act on the members of the set of bosons. $\bar{Q}_{\dot{B}}$ transforms bosons to fermions and Q_A does the reverse mapping. If P_μ is one-to-one, then so is $\{Q_A, \bar{Q}_{\dot{B}}\} = 2\sigma^\mu{}_{A\dot{B}}P_\mu$. Thus there must be an equal number in both sets.[21]

Let us expand on the two simplest examples. For $j = 0$ the Clifford vacuum $|\Omega\rangle$ has $s = 0$ and is a bosonic state. There are two states $\bar{Q}^{\dot{A}}|\Omega\rangle$ with $s = \frac{1}{2}$ and $s_3 = \mp\frac{1}{2}$ and one state $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$ with $s = 0$ and $s_3 = 0$. In total there are two scalar particles and two spin-$\frac{1}{2}$ fermions. Note that all these particles have the same mass. We will later refer to this set of states as the **scalar superfield**.

For $j = \frac{1}{2}$ we have two Clifford vacua $|\Omega\rangle$ with $j_3 = \pm\frac{1}{2}$, and with $s = \frac{1}{2}$ and $s_3 = \pm\frac{1}{2}$ (thus they are fermionic states). For the moment we label them as $|\Omega; \frac{1}{2}\rangle$ and $|\Omega; -\frac{1}{2}\rangle$. From each of these we can construct two further fermion states $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega; \pm\frac{1}{2}\rangle$ with $s_3 = \mp\frac{1}{2}$. In addition to this we have the states $\bar{Q}^{\dot{1}}|\Omega; \frac{1}{2}\rangle$ and $\bar{Q}^{\dot{2}}|\Omega; -\frac{1}{2}\rangle$ with $s_3 = 0$, the state $\bar{Q}^{\dot{2}}|\Omega; \frac{1}{2}\rangle$ with $s_3 = 1$, and the state $\bar{Q}^{\dot{1}}|\Omega; -\frac{1}{2}\rangle$ has $s_3 = -1$. Together these states can form two fermions with $s = \frac{1}{2}$ and $s_3 = \pm\frac{1}{2}$, one massive vector particle with $s = 1$, and $s_3 = 1, 0, -1$, and one scalar with $s = 0$.[22]   We will later refer to this set of states as the **vector superfield**.

Exercise: What are the states for $j = 1$?

We should use the term particle here very lightly since the states we have found are spinor states. A real Dirac fermion can only be described by a $j = 0$ representation and a

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[20]The same can easily be shown for $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$.

[21]Observe that this tells us that there must be an equal number of states in both sets, not particles.

[22]For massless particles, $m = 0$, we can form a vector particle with $s_3 = \pm 1$ and one extra scalar.

complex conjugate representation, thus having four degrees of freedom (d.o.f.). In field theory calculations, when the fermion is on-shell, two of these are eliminated in the Dirac equation, thus we get the expected two d.o.f. for a fermion.

Chapter 3

Superspace

In this chapter we will introduce a very handy notation for considering supersymmetry transformations effected by the superalgebra, or, more correctly, the elements of the super-Poincaré group. This is called superspace, and allows us to define so-called superfields. In order to do this we first need to know a little about the properties of Grassman numbers.

3.1 Superspace calculus

Grassman numbers θ are numbers that anti-commute with each others but not with ordinary numbers. We will here use four such numbers, and in addition we want to place them in Weyl spinors:[1]

$$\{\theta^A, \theta^B\} = \{\theta^A, \bar{\theta}^{\dot{B}}\} = \{\bar{\theta}^{\dot{A}}, \theta^B\} = \{\bar{\theta}^{\dot{A}}, \theta^{\dot{B}}\} = 0.$$

From this we get the relationships:[2]

$$\theta_A^2 \;\; = \;\; \theta_A \theta_A = -\theta_A \theta_A = 0, \tag{3.1}$$

$$\theta^2 \;\; \equiv \;\; \theta\theta \equiv \theta^A \theta_A = -2\theta_1 \theta_2, \tag{3.2}$$

$$\bar{\theta}^2 \;\; \equiv \;\; \bar{\theta}\bar{\theta} \equiv \bar{\theta}_{\dot{A}} \bar{\theta}^{\dot{A}} = 2\bar{\theta}^{\dot{1}} \bar{\theta}^{\dot{2}}. \tag{3.3}$$

Notice that if we have a function f of a Grassman number, say θ_A, then the all-order expansion of that function in terms of θ_A, is

$$f(\theta_A) = a_0 + a_1 \theta_A, \tag{3.4}$$

there simply are no more terms because of (3.1).

We now need to define differentiation and integration on these numbers in order to create a calculus for them.

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[1]We can already see how this can be handy: if we consistently use $\theta^A Q_A$ and $\bar{\theta}_{\dot{A}} \bar{Q}^{\dot{A}}$ instead of only Q_A and $\bar{Q}^{\dot{A}}$ in Eqs. (2.22)–(2.25) we can actually rewrite the superalgebra as an ordinary Lie algebra because of these commutation properties.

[2]There is no summation implied in the first line.

Definition: We define differentiation by:[a]

$$\partial_A \theta^B \equiv \frac{\partial}{\partial \theta^A} \theta^B \equiv \delta_A{}^B,$$

with a product rule

$$\begin{aligned}
\partial_A(\theta^{B_1}\theta^{B_2}\theta^{B_3}\dots\theta^{B_n}) &\equiv (\partial_A\theta^{B_1})\theta^{B_2}\theta^{B_3}\dots\theta^{B_n} \\
&\quad -\theta^{B_1}(\partial_A\theta^{B_2})\theta^{B_3}\dots\theta^{B_n} \\
&\quad +\dots+(-1)^{n-1}\theta^{B_1}\theta^{B_2}\dots(\partial_A\theta^{B_n}).
\end{aligned} \qquad (3.5)$$

---

[a]Note that this has no infinitesimal interpretation.

Definition: We define integration by $\int d\theta_A \equiv 0$ and $\int d\theta_A \theta_A \equiv 1$ and we demand linearety:

$$\int d\theta_A[af(\theta_A)+bg(\theta_A)] \equiv a\int d\theta_A f(\theta_A) + b\int d\theta_A g(\theta_A).$$

This has one surprising property. If we take the integral of (3.4) we get:

$$\int d\theta_A f(\theta_A) = a_1 = \partial^A f(\theta_A),$$

meaning that differentiation and integration has the same effect on Grassman numbers.

To integrate over multiple Grassman numbers we define volume elements for the Weyl spinors

Definition:

$$d^2\theta \equiv -\frac{1}{4}d\theta^A d\theta_A,$$

$$d^2\bar\theta \equiv -\frac{1}{4}d\bar\theta_{\dot A} d\bar\theta^{\dot A},$$

$$d^4\theta \equiv d^2\theta d^2\bar\theta.$$

This means that

$$\int d^2\theta \; \theta\theta = 1$$

$$\int d^2\bar\theta \; \bar\theta\bar\theta = 1$$

$$\int d^4\theta \; (\theta\theta)(\bar\theta\bar\theta) = 1$$

Delta functions of Grassmann variables are given by:

$$\delta(\theta_A) = \theta_A$$

$$\delta^2(\theta_A) = \theta\theta$$

$$\delta^2(\bar{\theta}^{\dot{A}}) = \bar{\theta}\bar{\theta}$$

and these functions satisfy (just as the usual definition of delta functions):

$$\int d\theta_A f(\theta_A)\delta(\theta_A) = f(0).$$

3.2 Superspace definition (Salam & Strathdee [5])

Superspace is a coordinate system where supersymmetry transformations are manifest, in other words, the action of elements in the super-Poincaré group (SP) based on the superalgebra are treated like Lorentz-transformations are in Minkowski space.

> **Definition**: **Superspace** is an eight-dimension manifold that can be constructed from the **coset space** of the super-Poincaré group (SP) and the Lorentz group (L), SP/L, by giving coordinates $z^\pi = (x^\mu, \theta^A, \bar{\theta}^{\dot{A}})$, where x^μ are the ordinary Minkowski coordinates, and where θ_A and $\bar{\theta}^{\dot{A}}$ are four Grassman (anti-commuting) numbers, being the parameters of the Q-operators in the algebra.

To see this we begin by writing a general element of SP, $g \in SP$, as[3]

$$g = \exp[-ix^\mu P_\mu + i\theta^A Q_A + i\bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}} - \frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}],$$

where x^μ, θ^A, $\bar{\theta}_{\dot{A}}$ and $\omega_{\rho\nu}$ constitute the parametrization of the group, and P_μ, Q_A, $\bar{Q}^{\dot{A}}$ and $M_{\rho\nu}$ are the generators. We can now parametrise SP/L simply by setting $\omega_{\mu\nu} = 0$.[4] The remaining parameters of SP/L then span superspace.

As we are physicists we also want to know the dimensions of our new parameters. To do this we first look at Eq. (2.23):

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2\sigma^\mu{}_{A\dot{B}}P_\mu$$

we know that P_μ has mass dimension $[P_\mu] = M$. This means that $[Q^2] = M$ and $[Q] = M^{\frac{1}{2}}$. In the exponential, all terms must have mass dimension zero to make sense. This means that $[\theta Q] = 0$, and therefore $[\theta] = M^{-\frac{1}{2}}$.

In order to show the effect of supersymmetry transformations, we begin by noting that any SP transformation can effectively be written in the following way:

$$L(a, \alpha) = \exp[-ia^\mu P_\mu + i\alpha^A Q_A + i\bar{\alpha}^{\dot{A}}\bar{Q}_{\dot{A}}],$$

---

[3]We hava already used this property, but this is what is formally called an **exponential map** of the Lie algebra to the Lie group. For matrix Lie groups this is simply the matrix exponential shown here. Technicaly this provides a **local cover** of the group around small values for the parameters.

[4]SP/L is not a coset group as defined previously, because L is not a normal subgroup of SP, but its parametrisation still forms a vector space which we call superspace.

because one can show that[5]

$$\exp\left[-\frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right]L(a,\alpha) = L(\Lambda a, S(\Lambda)\alpha)\exp\left[-\frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right], \tag{3.6}$$

i.e. all that a Lorentz boost does is to transform spacetime coordinates by $\Lambda(M)$ and Weyl spinors by $S(\Lambda(M))$, which is a spinor representation of $\Lambda(M)$. Thus, we can pick frames, do our thing with the transformation, and boost back to any frame we wanted. In addition, since P_μ commutes with all the Qs, when we speak of the supersymmetry transformation we usually mean just the transformation

$$\delta_S = \alpha^A Q_A + \bar{\alpha}_{\dot{A}}\bar{Q}^{\dot{A}}. \tag{3.7}$$

We can now find the transformation of superspace coordinates under a supersymmetry transformation, just as we have all seen the transformation of Minkowski coordinates under Lorentz transformations. The effect of $g_0 = L(a,\alpha)$ on a superspace coordinate $z^\pi = (x^\mu, \theta^A, \bar{\theta}_{\dot{A}})$ is defined by the mapping $z^\pi \to z'^\pi$ given by $g_0 e^{iz^\pi K_\pi} = e^{iz'^\pi K_\pi}$ where $K_\pi = (P_\mu, Q_A, \bar{Q}^{\dot{A}})$. We have[6]

$$\begin{aligned}
g_0 e^{iz^\pi K_\pi} &= \exp(-ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}})\exp(iz^\pi K_\pi) \\
&= \exp(-ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}} + iz^\pi K_\pi \\
&\quad -\frac{1}{2}[-ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}}, iz^\pi K_\pi] + \ldots)
\end{aligned}$$

Here we take a closer look at the commutator:[7]

$$\begin{aligned}
[\ ,\ ] &= [\alpha^B Q_B, \bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}}] + [\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}}, \theta^A Q_A] \\
&= -\alpha^B \bar{\theta}_{\dot{A}}\epsilon^{\dot{A}\dot{C}}\{Q_B, \bar{Q}_{\dot{C}}\} - \bar{\alpha}_{\dot{B}}\theta^A \epsilon^{\dot{B}\dot{C}}\{\bar{Q}_{\dot{C}}, Q_A\} \\
&= -2\alpha^B \bar{\theta}_{\dot{A}}\epsilon^{\dot{A}\dot{C}}\sigma^\mu{}_{B\dot{C}}P_\mu - \bar{\alpha}_{\dot{B}}\theta^A \epsilon^{\dot{B}\dot{C}}\sigma^\mu{}_{A\dot{C}}P_\mu \\
&= (-2\alpha^B \bar{\theta}^{\dot{C}}\sigma^\mu{}_{B\dot{C}} - 2\bar{\alpha}^{\dot{C}}\theta^A \sigma^\mu{}_{A\dot{C}})P_\mu
\end{aligned}$$

We can relabel $B = A$ and $\dot{C} = \dot{A}$ which leads to

$$-\frac{1}{2}[\ ,\ ] = (\alpha^A \sigma^\mu{}_{A\dot{A}}\bar{\theta}^{\dot{A}} - \theta^A \sigma^\mu{}_{A\dot{A}}\bar{\alpha}^{\dot{A}})P_\mu.$$

The commutator is proportional with P_μ, and will therefore commute with all operators, in particular the higher terms in the Campbell-Baker-Hausdorff expansion, meaning that the series reduces to

$$\begin{aligned}
&g_0 e^{iZ^\pi K_\pi} \\
&= \exp[i(-x^\mu - a^\mu + i\alpha^A \sigma^\mu{}_{A\dot{A}}\bar{\theta}^{\dot{A}} - i\theta^A \sigma^\mu{}_{A\dot{A}}\bar{\alpha}^{\dot{A}})P_\mu + i(\theta^A + \alpha^A)Q_A + i(\bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}})\bar{Q}^{\dot{A}}].
\end{aligned}$$

---

[5]Fortunately we are not going to do this because it is messy, but it can be done using the algebra of the group and the series expansion of the exponential function. Note, however, that the proof rests on the Ps and Qs forming a closed set, which we saw in the algebra Eqs. (2.22)–(2.25).

[6]Here we use Campbell-Baker-Hausdorff expansion $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}-\frac{1}{2}[\hat{A},\hat{B}]+\cdots}$ where the next term contains commutators of the first commutator and the operators $\hat{A}$ and $\hat{B}$.

[7]Using that P_μ commutes with all elements in the algebra, as well as $[\theta^A Q_A, \xi^B Q_B] = \theta^A \xi^B \{Q_A, Q_B\} = 0$, and the same for $\bar{Q}^{\dot{B}}$.

So superspace coordinates transform under supersymmetry transformations as:

$$(x^\mu, \theta^A, \bar{\theta}_{\dot{A}}) \rightarrow f(a^\mu, \alpha^A, \bar{\alpha}_{\dot{A}}) = (x^\mu + a^\mu - i\alpha^A \sigma^\mu{}_{A\dot{A}} \bar{\theta}^{\dot{A}} + i\theta^A \sigma^\mu{}_{A\dot{A}} \bar{\alpha}^{\dot{A}}, \theta^A + \alpha^A, \bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}}). \quad (3.8)$$

As a by-product we can now write down a differential representation for the supersymmetry generators by applying the standard expression for the generators X_i of a Lie algebra, given the functions f_π for the transformation of the parameters:

$$X_j = \frac{\partial f_\pi}{\partial a_j} \frac{\partial}{\partial z_\pi}$$

which gives us:[8]

$$P_\mu = i\partial_\mu \quad (3.9)$$
$$iQ_A = -i(\sigma^\mu \bar{\theta})_A \partial_\mu + \partial_A \quad (3.10)$$
$$i\bar{Q}^{\dot{A}} = -i(\bar{\sigma}^\mu \theta)^{\dot{A}} \partial_\mu + \partial^{\dot{A}} \quad (3.11)$$

Exercise: Check that Eqs. (3.9)–(3.11) fulfil the superalgebra in Eqs. (2.22)–(2.24).

3.3 Covariant derivatives

Similar to the properties of covariant derivatives for gauge transformations in gauge theories, it would be nice to have a derivative that is invariant under supersymmetry transformations, *i.e.* commutes with supersymmetry operators. Obviously $P_\mu = i\partial_\mu$ does this, but more general covariant derivatives can be made.

Definition: The following covariant derivatives commute with supersymmetry transformations:

$$D_A \equiv \partial_A + i(\sigma^\mu \bar{\theta})_A \partial_\mu, \quad (3.12)$$
$$\bar{D}_{\dot{A}} \equiv -\partial_{\dot{A}} - i(\theta \sigma^\mu)_{\dot{A}} \partial_\mu. \quad (3.13)$$

These can be shown to satisfy relations that are useful in calculations:

$$\{D_A, D_B\} = \{\bar{D}_{\dot{A}}, \bar{D}_{\dot{B}}\} = 0 \quad (3.14)$$
$$\{D_A, \bar{D}_{\dot{B}}\} = -2\sigma^\mu_{A\dot{B}} P_\mu \quad (3.15)$$
$$D^3 = \bar{D}^3 = 0 \quad (3.16)$$
$$D^A \bar{D}^2 D_A = \bar{D}_{\dot{A}} D^2 \bar{D}^{\dot{A}} \quad (3.17)$$

From the covariant derivatives we can construct projection operators.

---

[8]We define the generators X_i as $-iP_\mu$, iQ_A and iQ_B respectively.

Definition: The operators

$$\pi_+ \ \equiv \ -\frac{1}{16\square}\bar{D}^2 D^2, \tag{3.18}$$

$$\pi_- \ \equiv \ -\frac{1}{16\square}D^2 \bar{D}^2, \tag{3.19}$$

$$\pi_T \ \equiv \ \frac{1}{8\square}\bar{D}_{\dot{A}} D^2 \bar{D}^{\dot{A}}, \tag{3.20}$$

with $\square \equiv \partial_\mu \partial^\mu$, are projection operators, *i.e.* they fulfill:

$$\pi^2_{\pm,T} \ = \ \pi_{\pm,T} \tag{3.21}$$

$$\pi_+ \pi_- \ = \ \pi_+ \pi_T = \pi_- \pi_T = 0 \tag{3.22}$$

$$1 \ = \ \pi_+ + \pi_- + \pi_T \tag{3.23}$$

3.4   Superfields

Definition: A **superfield** Φ is an operator valued function on superspace $\Phi(x,\theta,\bar{\theta})$.

We can expand any Φ in a power series in θ and $\bar{\theta}$. In general:[9]

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) \ = \ & f(x) + \theta^A \varphi_A(x) + \bar{\theta}_{\dot{A}}\bar{\chi}^{\dot{A}}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) \\
& + \theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta\theta\bar{\theta}_{\dot{A}}\bar{\lambda}^{\dot{A}}(x) + \bar{\theta}\bar{\theta}\theta^A \psi_A(x) + \theta\theta\bar{\theta}\bar{\theta}d(x).
\end{aligned} \tag{3.24}$$

The properties of the component fields of a superfield can be deduced from Φ being a Lorentz scalar. This is shown in Table 3.1

Component field	Type	d.o.f.
$f(x)$, $m(x)$, $n(x)$	Complex(pseudo) scalar	2
$\psi_A(x)$, $\varphi_A(x)$	Left-handed Weyl spinors	4
$\bar{\chi}^{\dot{A}}(x)$, $\bar{\lambda}^{\dot{A}}(x)$	Right-handed Weyl spinors	4
$V_\mu(x)$	Lorentz 4-vector	8
$d(x)$	Complex scalar	2

Table 3.1: Fields contained in a general superfield.

One can show (tedious) that under supersymmetry transformations these component fields transform linearly into each other, thus superfields are **representations** of the supersymmetry (super-Poincaré) algebra, albeit *highly reducible* representations![10] We can recover the

---

[9]Note that any superfield commutes with any other superfield, because all Grassmann numbers appear in pairs. Equation (3.24) can be shown to be closed under supersymmetry transformations, meaning that a superfield transforms into another superfield under the transformations of the previous section.

[10]Indeed they are linear representations since a sum of superfields is a superfield, and the differential supersymmetry operators act linearly.

known irreducible representations, see Section 2.7, by some rather *ad hoc* restrictions on the fields:[11]

$$
\begin{align}
\bar{D}_{\dot{A}}\Phi(x,\theta,\bar{\theta}) &= 0 \quad \text{(left-handed scalar superfield)} \tag{3.25}\\
D_A\Phi^\dagger(x,\theta,\bar{\theta}) &= 0 \quad \text{(right-handed scalar superfield)} \tag{3.26}\\
\Phi^\dagger(x,\theta,\bar{\theta}) &= \Phi(x,\theta,\bar{\theta}) \quad \text{(vector superfield)} \tag{3.27}
\end{align}
$$

Products of same-handed superfields are also superfields with the same handedness:

$$
\bar{D}_{\dot{A}}(\Phi_i\Phi_j) = (\bar{D}_{\dot{A}}\Phi_i)\Phi_j + \Phi_i(\bar{D}_{\dot{A}}\Phi_j) = 0
$$

This is important when creating a **superpotential**, the supersymmetric precursor to a full Lagrangian.[12]

Note that the projection operators that we defined in Section 3.3, $\pi_\pm$, project out left-/right-handed superfields, respectively, because:

$$
\bar{D}_{\dot{A}}\pi_+\Phi = D_A\pi_-\Phi^\dagger = 0.
$$

This is analogous to the familiar properties of $P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$ in field theory.

3.4.1 Scalar superfields

What is the connection of the scalar superfields to the $j = 0$ irreducible representation? We use a cute[13] trick: Change to the variable $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$. Then:

$$
\begin{align}
D_A &= \partial_A + 2i\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}}\frac{\partial}{\partial y^\mu}, \tag{3.28}\\
\bar{D}_{\dot{A}} &= -\partial_{\dot{A}}. \tag{3.29}
\end{align}
$$

This means that a field fulfilling $\bar{D}_{\dot{A}}\Phi = 0$ in the new set of coordinates must be independent of $\bar{\theta}$. Thus we can write:

$$
\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y),
$$

and looking at the field content we get the result in Table 3.2.

Component field	Type	d.o.f.
$A(x)$, $F(x)$	Complex scalar	2
$\psi(x)$	Left-handed Weyl spinors	4

Table 3.2: Fields contained in a left-handed scalar superfield.

We can undo the coordinate change and get:[14]

$$
\Phi(x,\theta,\bar{\theta}) = A(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu A(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x).
$$

---

[11]Note that the dagger here is part of the name of the field.

[12]Supersymmetry transformations can be shown to transform left-handed superfields into left-handed superfields and right-handed superfields into right-handed superfields.

[13]Here cute is used in the widest sense.

[14]Just by expanding the above in powers of θ and $\bar{\theta}$.

By doing the transformation $y^\mu \equiv x^\mu - i\theta\sigma^\mu\bar\theta$ we can show a similar field content for the right handed scalar superfield. The general form of a right handed scalar superfield is then:

$$\Phi^\dagger(x,\theta,\bar\theta) = A^*(x) - i(\theta\sigma^\mu\bar\theta)\partial_\mu A^*(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\Box A^*(x) + \sqrt{2}\bar\theta\bar\Psi(x) + \frac{i}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\Psi(x) + \bar\theta\bar\theta F^*(x).$$

These fields will not correspond directly to particle states. After applying the equations of motions (e.o.m.) the (auxillary) field $F(x)$ can be eliminated as it does not have any derivatives. The e.o.m. also eliminates two of the fermion d.o.f. and a Weyl spinor on its own cannot describe a Dirac fermion. When we construct particle representations we will take one left-handed scalar superfield and one different right-handed scalar superfield. These will form a fermion and two scalars (and their anti-particles). We see from (3.25) and (3.26) that if Φ is left handed, then $\Phi^\dagger$ is right handed and vice versa, the dagger now signifying hermitian conjugation.

3.4.2   Vector superfields

We take the general superfield and compare Φ and $\Phi^\dagger$. We see that the following is the structure of a general vector superfield:

$$\begin{aligned}
\Phi(x,\theta,\bar\theta) &= f(x) + \theta\varphi(x) + \bar\theta\bar\varphi(x) + \theta\theta m(x) + \bar\theta\bar\theta m^*(x) \\
&\quad + \theta\sigma^\mu\bar\theta V_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\lambda(x) + \theta\theta\bar\theta\bar\theta d(x).
\end{aligned}$$

and looking at the component fields we find the results in Table 3.3.

Component field	Type	d.o.f.
$f(x)$, $d(x)$	Real scalar field	1
$\varphi(x)$, $\lambda(x)$	Weyl spinors	4
$m(x)$	Complex scalar field	2
$V_\mu(x)$	Real Lorentz 4-vector	4

Table 3.3: Fields contained in a general vector superfield.

One example of a vector superfield is the product $V = \Phi^\dagger\Phi$ where we easily see that $V^\dagger = (\Phi^\dagger\Phi)^\dagger = \Phi^\dagger(\Phi^\dagger)^\dagger = \Phi^\dagger\Phi$. Note that sums and products of vector superfields are also vector superfields:

$$(V_i + V_j)^\dagger = V_i^\dagger + V_j^\dagger = V_i + V_j,$$

and

$$(V_i V_j)^\dagger = V_j^\dagger V_i^\dagger = V_i V_j.$$

You may now be a little suspicious that this vector superfield does not correspond to the promised degrees of freedom in the $j = \frac{1}{2}$ representation of the superalgebra. Gauge-freedom comes to the rescue.

3.5   Supergauge

We begin with the definition of a (super) gauge transformation on a vector superfield[15]

---

[15] And promise we will get back to the corresponding definition for a scalar superfield.

Definition: Given a vector superfield $V(x, \theta, \bar{\theta})$, we define the **abelian supergauge-transformation** as

$$V(x, \theta, \bar{\theta}) \rightarrow V'(x, \theta, \bar{\theta}) \;=\; V(x, \theta, \bar{\theta}) + \Phi(x, \theta, \bar{\theta}) + \Phi^{\dagger}(x, \theta, \bar{\theta})$$
$$\equiv\; V(x, \theta, \bar{\theta}) + i(\Lambda(x, \theta, \bar{\theta}) - \Lambda^{\dagger}(x, \theta, \bar{\theta}))$$

where the parameter of the transformation Φ (or Λ) is a scalar superfield.

One can show that under supergauge transformations the vector superfield components transform as:

$$
\begin{aligned}
f(x) &\rightarrow& f'(x) &= f(x) + A(x) + A^*(x) & \text{(3.30)}\\
\varphi(x) &\rightarrow& \varphi'(x) &= \varphi(x) + \sqrt{2}\psi(x) & \text{(3.31)}\\
m(x) &\rightarrow& m'(x) &= m(x) + F(x) & \text{(3.32)}\\
V_{\mu}(x) &\rightarrow& V_{\mu}'(x) &= V_{\mu}(x) + i\partial_{\mu}(A(x) - A^*(x)) & \text{(3.33)}\\
\lambda(x) &\rightarrow& \lambda'(x) &= \lambda(x) & \text{(3.34)}\\
d(x) &\rightarrow& d'(x) &= d(x) & \text{(3.35)}
\end{aligned}
$$

Exercise: Show the vector superfield component field transformation properties, using the redefinitions:

$$\lambda(x) \rightarrow \lambda(x) + \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\varphi}(x)$$

$$d(x) \rightarrow d(x) - \frac{1}{4}\Box f(x)$$

Notice that from the above the standard field strength for a vector field, $F_{\mu\nu} = \partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu}$, is supergauge invariant. With the newfound freedom of gauge invariance we can choose component fields of Φ to eliminate some remaining reducibility.

Definition: The **Wess-Zumiono (WZ) gauge** is a supergauge transformation of a vector superfield by a scalar superfield with

$$
\begin{aligned}
\psi(x) &=& -\frac{1}{\sqrt{2}}\varphi(x), & \text{(3.36)}\\
F(x) &=& -m(x), & \text{(3.37)}\\
A(x) + A^*(x) &=& -f(x). & \text{(3.38)}
\end{aligned}
$$

A vector superfield in the WZ gauge can be written:

$$V_{WZ}(x, \theta, \bar{\theta}) = (\theta\sigma^{\mu}\bar{\theta})[V_{\mu}(x) + i\partial_{\mu}(A(x) - A^*(x))] + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}d(x),$$

which contains one real scalar field d.o.f., three gauge field d.o.f. and four fermion d.o.f., cor-

responding to the representation $j = \frac{1}{2}$.[16] The WZ gauge is particularly convenient because:

$$V_{WZ}^2 = \frac{1}{2}\theta\theta\bar\theta\bar\theta[V_\mu(x) + i\partial_\mu(A(x) - A^*(x))][V^\mu(x) + i\partial^\mu(A(x) - A^*(x))]$$

and

$$V_{WZ}^3 = 0,$$

so that

$$e^{V_{WZ}} = 1 + V_{WZ} + \frac{1}{2}V_{WZ}^2.$$

---

[16]Note that supersymmetry transformations break this gauge.

Chapter 4

Construction of a low-energy supersymmetric Lagrangian

We would now like to construct a model that is invariant under supersymmetry transformation, much in the same way that the Standard Model is invariant under Poincaré transformations.

4.1 Supersymmetry invariant Lagrangians and actions

As should be well known the **action**

$$S \equiv \int_R d^4x \, \mathcal{L}, \tag{4.1}$$

is invariant under supersymmetry transformations if this transforms the Lagrangian by a total derivative term $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial^\mu f(x)$, where $f(x) \to 0$ on $S(R)$ (the surface of the integration region R). The question then becomes: how can we construct a Lagrangian from superfields with this property?

We can show that the highest order component fields in θ and $\bar{\theta}$ of a superfield always transform in this way, *e.g.* for the general superfield the highest order component field $d(x)$ transforms under the supersymmetry transformation

$$\delta_s d(x) = d'(x) - d(x),$$

as

$$\delta_s d(x) = \frac{i}{2}(\partial_\mu \psi(x)\sigma^\mu \bar{\alpha} - \partial_\mu \bar{\lambda}(x)\sigma^\mu \alpha),$$

where the constant α is the supersymmetry transformation parameter.[1] These highest power component can be isolated by using the projection property of integration in Grassman calculus so that

$$S = \int_R d^4x \int d^4\theta \, \mathcal{L},$$

where $\mathcal{L}$ is a function of superfields, is guaranteed to be supersymmetry invariant. Note that this constitutes a redefinition of what we mean by $\mathcal{L}$, and one should be careful when counting

---

[1]Note that this is a **global** SUSY transformation. Replacing $\alpha \to \alpha(x)$ gives a **local** SUSY transformation, which, it turns out, leads to supergravity.

the dimension of terms.[2] We now have a generic form for the supersymmetry Lagrangian, where the indices indicate the highest power of θ in the term:

$$\mathcal{L} = \mathcal{L}_{\theta\theta\bar{\theta}\bar{\theta}} + \theta\theta\mathcal{L}_{\bar{\theta}\bar{\theta}} + \bar{\theta}\bar{\theta}\mathcal{L}_{\theta\theta}.$$

The requirement of renormalizability puts further restrictions on the fields in $\mathcal{L}$. We can at most have three powers of scalar superfields, for details see *e.g.* Wess & Bagger [6]. Since the action must be real, the (almost) most general supersymmetry Lagrangian that can be written in terms of scalar superfields is:

$$\mathcal{L} = \Phi_i^\dagger \Phi_i + \bar{\theta}\bar{\theta}W[\theta] + \theta\theta W[\Phi^\dagger].$$

Here the first term is called the **kinetic term**[3], and W is the **superpotential**

$$W[\Phi] = g_i\Phi_i + m_{ij}\Phi_i\Phi_j + \lambda_{ijk}\Phi_i\Phi_j\Phi_k. \tag{4.2}$$

This means that to specify a supersymmetric Lagrangian we only need to specify the superpotential. Dimension counting for the couplings give $[g_i] = M^2$, $[m_{ij}] = M$ and $[\lambda_{ijk}] = 1$. Notice also that m_{ij} and λ_{ijk} are symmetric.

4.2   Abelian gauge theories

We would ultimately like to have a gauge theory like that of the SM, so we start with an abelian warm-up, by finally definig what we mean by an (abelian) supergauge transformation on a scalar superfield.

> **Definition**: The $U(1)$ **(super)gauge transformation** (local or global) on left handed scalar superfields is defined as:
>
> $$\Phi_i \to \Phi_i' = e^{-i\Lambda q_i}\Phi_i$$
>
> where q_i is the $U(1)$ charge of Φ_i and Λ, or $\Lambda(x)$, is the parameter of the gauge transformation.

For the definition to make sense Φ_i' must be a left-handed scalar superfield, thus

$$\bar{D}_{\dot{A}}\Phi_i' = 0,$$

and this requires:

$$\begin{aligned}
\bar{D}_{\dot{A}}\Phi_i' &= \bar{D}_{\dot{A}}e^{-i\Lambda q_i}\Phi_i = e^{-i\Lambda q_i}\bar{D}_{\dot{A}}\Phi_i - iq_i(\bar{D}_{\dot{A}}\Lambda)e^{-i\Lambda q_i}\Phi_i \\
&= -iq_i(\bar{D}_{\dot{A}}\Lambda)\Phi_i' = 0.
\end{aligned}$$

Thus we must have $\bar{D}_{\dot{A}}\Lambda = 0$, which by definition means that Λ itself is a left-handed superfield. This is of course completely equivalent for right-handed scalar fields.

---

[2]Looking at the mass dimensions we have, since $\int d\theta\,\theta = 1$ from superspace calculus (see Section 3.1), $[\theta] = M^{-1/2}$ which leads to $[\int d\theta] = M^{1/2}$. We then have $[\int d^4\theta] = M^2$. Since we must have $[\int d^4\theta\,\mathcal{L}] = M^4$ for the action to be dimensionless, we need $[\mathcal{L}] = M^2$.

[3]The constant in front can always be chosen to be one because we can rescale the whole Lagrangian. Notice that the kinetic terms are vector superfields.

We will of course now require not only a supersymmetry invariant Lagrangian, but also a gauge invariant Lagrangian. Let us first look at the transformation of the superpotential W under the gauge transformation:

$$W[\Phi] \to W[\Phi'] = g_i e^{-i\Lambda q_i}\Phi_i + m_{ij}e^{-i\Lambda(q_i+q_j)}\Phi_i\Phi_j + \lambda_{ijk}e^{-i\Lambda(q_i+q_j+q_k)}\Phi_i\Phi_j\Phi_k$$

For $W[\Phi] = W[\Phi']$ we must have:

$$
\begin{align}
g_i &= 0 \ \text{ if } q_i \neq 0 \tag{4.3}\\
m_{ij} &= 0 \ \text{ if } \ q_i + q_j \neq 0 \tag{4.4}\\
\lambda_{ijk} &= 0 \ \text{ if } \ q_i + q_j + q_k \neq 0 \tag{4.5}
\end{align}
$$

This puts great restrictions on the form of the superpotential and the charge assignments of the superfields (as in ordinary gauge theories). What then about the kinetic term?

$$\Phi_i^\dagger\Phi_i \to \Phi_i^\dagger e^{i\Lambda^\dagger q_i}e^{-i\Lambda q_i}\Phi_i = e^{i(\Lambda^\dagger - \Lambda)q_i}\Phi_i^\dagger\Phi_i.$$

As in ordinary gauge theories we can introduce a **gauge compensating vector (super)field** V with the appropriate gauge transformation to make the kinetic term invariant under supersymmetry transformations. We can write the kinetic term as $\Phi_i^\dagger e^{q_i V}\Phi_i$, which gives us:

$$\Phi_i^\dagger e^{q_i V}\Phi_i \to \Phi_i^\dagger e^{i\Lambda^\dagger q_i}e^{q_i(V+i\Lambda-i\Lambda^\dagger)}e^{-i\Lambda q_i}\Phi_i = \Phi_i^\dagger e^{q_i V}\Phi_i$$

This definition of gauge transformation can be shown to recover the SM **minimal coupling** for the component fields through the covariant derivative

$$D_\mu^i = \partial_\mu - \frac{i}{2}q_i V_\mu,$$

where V_μ is the vector component field of the vector superfield.

In case you were worried: we can use the WZ gauge to show that the new kinetic term $\Phi_i^\dagger e^{q_i V}\Phi_i$ has no term with dimension higher then four, and is thus renormalizable.

4.3   Non-Abelian gauge theories

How do we extend the above to deal with much more complicated non-abelian gauge theories? Let us take a group G with the Lie algrabra of group generators t_a that fullfil

$$[t_a, t_b] = if_{ab}{}^c t_c, \tag{4.6}$$

where $f_{ab}{}^c$ are the structure constants. For an element g in the group G we want to write down a unitary[4] representation $U(g)$ that transforms a scalar superfield Ψ by $\Psi \to \Psi' = U(g)\Psi$. With an exponential map we can write the representation as $U(g) = e^{i\lambda^a t_a}$, as you may perhaps have expected.[5] Thus, we simply copy the abelian structure (as in ordinary gauge theories), and transform superfields as

$$\Psi \to \Psi' = e^{-iq\Lambda^a t_a}\Psi,$$

---

[4]By unitary we mean, as usual, that $U^\dagger = U^{-1}$ so that $U^\dagger U = 1$.
[5]Since we demanded a unitary representation the generators t_a must be hermitian.

where q is the charge of Ψ under G.[6] Again we can easily show that we must require that the Λ^a are left-handed scalar superfields for Ψ to transform to a left-handed scalar superfield.

For the superpotential to be invariant we must now have:

$$
\begin{align}
g_i &= 0 \quad \text{if} \quad g_i U_{ir} \neq g_r \tag{4.7}\\
m_{ij} &= 0 \quad \text{if} \quad m_{ij} U_{ir} U_{js} \neq m_{rs} \tag{4.8}\\
\lambda_{ijk} &= 0 \quad \text{if} \quad \lambda_{ijk} U_{ir} U_{js} U_{kt} \neq \lambda_{rst} \tag{4.9}
\end{align}
$$

where the indices on U are its matrix indices. We also want a similar construction for the kinetic terms as for abelian gauge theories, $\Psi^\dagger e^{qV^a T_a} \Psi$, to be invariant under non-abelian gauge transformations.[7] Now

$$
\Psi^\dagger e^{qV^a T_a} \Psi \rightarrow \Psi'^\dagger e^{qV'^a T_a} \Psi' = \Psi^\dagger e^{iq\Lambda^{a\dagger} T_a} e^{qV'^a T_a} e^{-iq\Lambda^a T_a} \Psi,
$$

so we have to require that the vector superfield V transforms as:

$$
e^{qV'^a T_a} = e^{-iq\Lambda^{a\dagger} T_a} e^{qV^a T_a} e^{iq\Lambda^a T_a}. \tag{4.10}
$$

When we look at this as an infinitesimal transformation in Λ we can show that

$$
V'^a = V^a + i(\Lambda^a - \Lambda^{a\dagger}) - \frac{1}{2} q f_{bc}{}^a V^b (\Lambda^{c\dagger} + \Lambda^c) + \mathcal{O}(\Lambda^2),
$$

which reduces to the abelian definition for abelian groups. If we look at the component vector fields, V_μ^a, these transform as for the standard gauge theory non-abelian

$$
V_\mu^a \rightarrow V_\mu'^a = V_\mu^a + i\partial_\mu(\Lambda^a - \Lambda^{a*}) - q f_{bc}{}^a V_\mu^b (\Lambda^c + \Lambda^{c*}),
$$

in the adjoint representation of the gauge group.[8]

The supergauge transformations of vector superfields can be written more efficiently in a representation independent way as

$$
e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda},
$$

and the inverse transformation is then given by

$$
e^{-V'} = e^{-i\Lambda} e^{-V} e^{i\Lambda^\dagger},
$$

where $\Lambda \equiv q\Lambda^a T_a$ and $V \equiv qV^a T_a$, such that $e^V e^{-V} = e^{V'} e^{-V'} = 1$.[9]

---

[6] At this point can choose a representation different from the fundamental, reflected in a different choice for t_a. Since we are almost exclusively interested in groups defined by a matrix representation $U(g)$ will be a matrix with dimension fixed by the dimension chosen for the representation.

[7] We have chosen some specific representation T_a of the generators t_a of the Lie algebra (4.6).

[8] This is independent of our choice of representation for the gauge group for the supergauge transformation.

[9] Notice that despite the non-commutative nature of the matrices involved, the identity $e^A e^{-A} = 1$ holds.

4.4 Supersymmetric field strength

There is one missing type of term for the supersymmetric Lagrangian, namely field strength terms, *e.g.* terms to describe the electromagetic field strength.

> **Definition**: **Supersymmetric field strength** is defined by the spinor (matrix) scalar superfields given by
> $$W_A \equiv -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_A e^V,$$
> and
> $$\bar{W}_{\dot{A}} \equiv -\frac{1}{4}DDe^{-V}\bar{D}_{\dot{A}}e^V.$$

We can show that W_A is a left-handed superfield and that $\text{Tr}[W^A W_A]$ (and $\text{Tr}[\bar{W}_{\dot{A}}\bar{W}^{\dot{A}}]$) is supergauge invariant and potential terms in the supersymmetry Lagrangian. Firstly

$$\bar{D}_{\dot{A}}W_A = -\frac{1}{4}\bar{D}_{\dot{A}}\bar{D}\bar{D}e^{-V}D_A e^V = 0,$$

because from Eq. (3.16) $\bar{D}^3 = 0$. Under a supergaugetransformation we have:

$$
\begin{aligned}
W_A \to W_A' &= -\frac{1}{4}\bar{D}\bar{D}e^{-i\Lambda}e^{-V}e^{i\Lambda^\dagger}D_A e^{-i\Lambda^\dagger}e^V e^{i\Lambda} \\
(\bar{D}_{\dot{A}}\Lambda = 0) &= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}e^{i\Lambda^\dagger}D_A e^{-i\Lambda^\dagger}e^V e^{i\Lambda} \\
(D_A\Lambda^\dagger = 0) &= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}D_A e^V e^{i\Lambda} \\
&= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}[(D_A e^V)e^{i\Lambda} + e^V(D_A e^{i\Lambda})] \\
&= e^{-i\Lambda}W_A e^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}D_A e^{i\Lambda}. \quad (4.11)
\end{aligned}
$$

We are free to add zero to (4.11) in the form of $-\frac{1}{4}e^{-i\Lambda}\bar{D}D_A\bar{D}e^{i\Lambda} = 0$,[10] giving

$$
\begin{aligned}
W_A' &= e^{-i\Lambda}W_A e^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\bar{D}\{\bar{D}, D_A\}e^{i\Lambda} \\
&= e^{-i\Lambda}W_A e^{i\Lambda} + \frac{1}{2}e^{-i\Lambda}\bar{D}_{\dot{A}}\sigma^\mu{}_{A\dot{B}}\epsilon^{\dot{A}\dot{B}}P_\mu e^{i\Lambda} \\
&= e^{-i\Lambda}W_A e^{i\Lambda},
\end{aligned}
$$

where we have used Eq. (3.15) to replace the anti-commutator. This means that the trace is gauge invariant:

$$
\begin{aligned}
\text{Tr}[W'^A W_A'] &= \text{Tr}[e^{-i\Lambda}W^A e^{i\Lambda}e^{-i\Lambda}W_A e^{i\Lambda}] \\
&= \text{Tr}[e^{i\Lambda}e^{-i\Lambda}W^A W_A] = \text{Tr}[W^A W_A].
\end{aligned}
$$

---

[10] Which is zero because Λ is a left-handed scalar superfield, $\bar{D}_{\dot{A}}\Lambda = 0$.

If we expand W_A in the component fields we find, as we might have hoped, that it contains the ordinary field strength tensor:

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + q f_{bc}{}^a V_\mu^b V_\mu^c$$

and that the trace indeed contains terms with $F_{\mu\nu}^a F^{\mu\nu a}$.

4.5   The (almost) complete supersymmetric Lagrangian

We can now write down the Lagrangian for a supersymmetric theory with (possibly) non-abelian gauge groups:[11]

$$\mathcal{L} = \Phi^\dagger e^V \Phi + \delta^2(\bar\theta) W[\Phi] + \delta^2(\theta) W[\Phi^\dagger] + \frac{1}{2T(R)} \delta(\bar\theta) \mathrm{Tr}[W^A W_A], \qquad (4.12)$$

where $T(R)$ is the Dynkin index that appears to correctly normalize the energy density for the chosen representation R of the gauge group. Note that since W_A is spanned by T_a for a given representation, we can write $W_A = W_A^a T_a$. Then

$$\mathrm{Tr}[W^A W_A] = W^{aA} W_A^b \mathrm{Tr}[T_a T_b] = W^{Aa} W_A^b \delta_{ab} T(R) = T(R) W^{aA} W_A^a. \qquad (4.13)$$

> **Exercise**: Write down the action of a supersymmetric field theory (without gauge transformations) in terms of component fields and show that it contains no kinetic terms for the $F_i(x)$ fields. Then show how they can be eliminated by the equations of motion. Challenge: Repeat for a gauge theory (here $d(x)$ can be eliminated).

4.6   Spontaneous supersymmetry breaking

As we have seen above, supersymmetry predicts scalar partner particles with the same mass as the known fermions (and new fermions for the known vectors). These, somewhat unfortunately, contradict experiment by not existing. In the SM we have a similar problem: the vector bosons should remain massless under the gauge symmetry of the model. Yet, they are observed to be very massive. This is solved with the introduction of the Higgs mechanism and **spontaneous symmetry breaking** in the **scalar potential**.[12] The idea is that while there is a symmetry of the Lagrangian (in the SM the gauge symmetry), this may not be a symmetry of the vacuum state, thereby allowing the properties of the vacuum to supply the masses. Would it not be great if we could have *spontaneous symmetry breaking* in order to break supersymmetry this way and boost the masses of supersymmetric particles beyond current limits?

---

[11]Note that there is no hermitian conjugate of the trace term, and an odd normalisation. This is because the term can be proven to be real, although this is sometimes overlooked in the literature.

[12]The potential of the Lagrangian are those terms not containing derivatives of the fields (kinetic terms). The scalar potential are such terms that contain only scalar fields.

From the exercise in the previous section we can see that the Lagrangian of (4.12) written in terms of component field contains no kinetic (derivative) terms for the $F(x)$ scalar fields. These are then what we call **auxilary fields** and can be eliminated by the e.o.m. we get from solving the Euler-Lagrange equation for this field:[13]

$$\frac{\partial \mathcal{L}}{\partial F_i^*(x)} = F_i(x) + W_i^* = 0,$$

where

$$W_i \equiv \frac{\partial W[A_1, ..., A_n]}{\partial A_i}. \tag{4.15}$$

This allows us to rewrite the action as (ignoring gauge interactions):

$$S = \int d^4x \{ i\partial_\mu \bar{\psi}_i \sigma^\mu \psi_i - A_i^* \Box A_i - \frac{1}{2} W_{ij} \psi_i \psi_j - \frac{1}{2} W_{ij}^* \bar{\psi}_i \bar{\psi}_j - |W_i|^2 \}$$

with[14]

$$W_{ij} \equiv \frac{\partial^2 W[A_1, ..., A_n]}{\partial A_i \partial A_j}. \tag{4.16}$$

Thus the scalar potential of the Lagrangian is

$$V(A_i, A_i^*) = \sum_{i=1}^n \left| \frac{\partial W[A_1, ..., A_n]}{\partial A_i} \right|^2. \tag{4.17}$$

In the SM figuring out a scalar potential that breaks $SU(2)_L \times U(1)_Y$ is a little messy. In supersymmetry the argument goes like this: First, notice that we can write the supersymmetric Hamiltonian as

$$H = \frac{1}{4}(Q_1 \bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}} Q_1 + Q_2 \bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}} Q_2).$$

To see this, consider

$$\begin{aligned} \{Q_A, \bar{Q}_{\dot{B}}\} \bar{\sigma}^{\nu \dot{B}A} &= 2\sigma^\mu{}_{A\dot{B}} \bar{\sigma}^{\nu \dot{B}A} P_\mu \\ &= 2\,\mathrm{Tr}[\sigma^\mu \bar{\sigma}^\nu] P_\mu \\ &= 4g^{\mu\nu} P_\mu = 4P^\nu. \end{aligned}$$

Now,

$$\begin{aligned} H &= P^0 = \frac{1}{4} \{Q_A, \bar{Q}_{\dot{B}}\} \bar{\sigma}^{0\dot{B}A} \\ &= \frac{1}{4}(Q_1 \bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}} Q_1 + Q_2 \bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}} Q_2). \end{aligned}$$

---

[13]We remind the reader that the Euler-Lagrange equation for a field ϕ is the result of minimizing the action and is given in terms of the Lagrangian as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{4.14}$$

[14]This is called the **fermionic mass matrix**.

As discussed in Section 2.5 we have $Q_A^\dagger = \bar{Q}_{\dot{A}}$. Thus the Hamiltonian is **semipositive definite**, *i.e.* $\langle \Psi | H | \Psi \rangle \geq 0$ for any state $|\Psi\rangle$.

Imagine now that there exists some lowest lying states (possibly degenerate), the ground state(s) $|0\rangle$, that have vanishing energy $\langle 0 | H | 0 \rangle = 0$. These are supersymmetric since, to fulfill the energy assumption, we must have

$$Q_A |0\rangle = \bar{Q}_{\dot{A}} |0\rangle = 0 \quad \text{for} \quad \forall A, \dot{A}, \tag{4.18}$$

and are thus invariant under the supersymmetry transformations given by (3.7)

$$\delta_S |0\rangle = (\alpha^A Q_A + \bar{\alpha}_{\dot{A}} \bar{Q}^{\dot{A}}) |0\rangle = 0. \tag{4.19}$$

This means that at this supersymmetric minimum of the potential the scalar potential must contribute zero

$$V(A, A^*) = 0 \quad \text{and thus} \quad \frac{\partial W}{\partial A_i} = 0.$$

Conversely, if the scalar potential does contribute in the vacuum (ground state) $|0\rangle$, meaning

$$\frac{\partial W}{\partial A_i} \neq 0 \quad \text{and thus} \quad V(A, A^*) > 0,$$

in the minimum of the potential for some A_i, then *supersymmetry must be broken*! As in the SM, the Lagrangian is still (super)symmetric, but $|0\rangle$ is not because (4.18) can no longer hold for all the Qs.

The **O'Raifeartaigh model** (1975) [7] is an example of a model that spontaneously breaks supersymmetry with three scalar superfields X, Y, Z, and the superpotential

$$W = \lambda Y Z + g X (Z^2 - m^2), \tag{4.20}$$

where λ, g and m are real non-zero parameters. The scalar potential is

$$\begin{aligned} V(A, A^*) &= \left| \frac{\partial W}{\partial A_X} \right|^2 + \left| \frac{\partial W}{\partial A_Y} \right|^2 + \left| \frac{\partial W}{\partial A_Z} \right|^2 \\ &= |g(A_Z^2 - m^2)|^2 + |\lambda A_Z|^2 + |\lambda A_Y + 2g A_X A_Z|^2, \end{aligned} \tag{4.21}$$

which can never be zero because setting $A_Z = 0$, which is needed for the second term, gives a non-zero contribution $g^2 m^4$ from the first term. Since the expectation value at the minimum that breaks supersymmetry is $\langle 0 | \frac{\partial W_i}{\partial A_i} | 0 \rangle$, and $F_i = \frac{\partial W_i}{\partial A_i}$, the condition for spontaneous ~~SUSY~~ (supersymmetry breaking) with the O'Raifertaigh mechanism can be written $\langle F_i \rangle \equiv \langle 0 | F_i(x) | 0 \rangle > 0$, hence it is given the name **F-term breaking**. In F-term breaking it is the vacuum expectation value (vev) of the auxilary field of a scalar superfield that supplies the breaking.

In a gauge theory, a similar mechanism is found by adding a term $\mathcal{L}_{FI} \sim 2kV$ where V is a vector superfield. The vev of the $d(x)$ auxiliary field will create a non-zero scalar potential.[15] This is called the **Fayet-Iliopolous model**, or **D-term breaking**.

---

[15]It is **always** the auxiliary fields fault!

4.7 Supertrace

Unfortunately, the above does not work in practice with all particles at a low energy scale. The problem is that at *tree level* the **supertrace**, STr, the weighted sum of eigenvalues of the mass matrix $\mathcal{M}$, can be shown to vanish, $\mathrm{STr}\,\mathcal{M}^2 = 0$.[16]

> **Definition**: The **supertrace** is given by
>
> $$\mathrm{STr}\,\mathcal{M}^2 \equiv \sum_s (-1)^{2s}(2s+1)\,\mathrm{Tr}\,M_s^2 \tag{4.22}$$
>
> where $\mathcal{M}$ is the mass matrix of the Lagrangian, s is the spin of particles and M_s is the mass matrix of all spin–s particles.

For a theory with only scalar superfields, with two fermionic and two bosonic degrees of freedom each, and with, respectively, mass matrices $M_{1/2}$ and M_0 after spontaneous supersymmetry breaking, this means that $\mathrm{Tr}\,\{M_0^2 - 2M_{1/2}^2\} = 0$, *i.e.* the sum of scalar particle masses (squared) is equal to the fermion masses (squared).[17] The consequence is that not all the scalar partners can be heavier than our known fermions.[18]

4.8 Soft breaking

What we can do instead is to add explicit supersymmetry breaking terms to the Lagrangian parametrizing our ignorance of the true (spontaneous) supersymmetry breaking on some higher scale $\sqrt{\langle F \rangle}$ that we do not have access to where the supertrace relation is fullfilled,[19] for which there are many alternatives in the literature, *e.g.*:

- Planck-scale Mediated Symmetry Breaking (PMSB)

- Gauge Mediated Symmetry Breaking (GMSB)

- Anomaly Mediated Symmetry Breaking (AMSB)

However, we cannot simply add arbitrary terms to the Lagrangian. The terms we can add are so-called **soft terms** with couplings of mass dimension one or higher. The dis-allowed terms with smaller mass dimension are terms that can lead to divergences in loop contributions to scalar masses (such as the Higgs) that are quadratic or worse (because of the high dimensionality of the fields in the loops). We will return to this issue in a moment. The allowed terms are in superfield notation as follows:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{soft}} = \ & -\frac{1}{4T(R)} M \theta\theta\bar{\theta}\bar{\theta}\,\mathrm{Tr}\{W^A W_A\} - \frac{1}{6} a_{ijk}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j\Phi_k \\
& -\frac{1}{2} b_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j - t_i\theta\theta\bar{\theta}\bar{\theta}\Phi_i + h.c. \\
& -m_{ij}^2\theta\theta\bar{\theta}\bar{\theta}\Phi_i^\dagger\Phi_j.
\end{aligned}
$$

---

[16] See Ferrara, Girardello and Palumbo (1979) [8].

[17] Remember that there are two scalar particles for each fermion.

[18] Strong coupling, meaning tree level is a bad approximation, may help, but life is still difficult.

[19] Remember that $[\Phi] = M$ and $[\theta] = M^{-\frac{1}{2}}$ so that the component field must have $[F] = M^2$.

Note that these terms are *not* supersymmetric. From the $\theta\theta\bar{\theta}\bar{\theta}$-factors we see that only the lowest order component fields of the superfields contribute. There are also some terms that are called "maybe-soft" terms:

$$\mathcal{L}_{\text{maybe}} = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}c_{ijk}\Phi_i^\dagger\Phi_j\Phi_k + \text{h.c.} \tag{4.23}$$

This last—oft ignored—type of term is soft as long as none of the scalar superfields is a singlet under all gauge symmetries. It is, however, quite difficult to get large values for c_{ijk} with spontaneous S̶U̶S̶Y̶. In the above terms we have not specified any gauge symmetry, which will, in the same way as it did for the superpotential, severely restrict the allowed terms. However, it turns out that soft-terms are responsible for most of the parameters in supersymmetric theories!

We can write the soft terms in terms of their component fields as[20]

$$\begin{aligned}
\mathcal{L}_{\text{soft}} &= -\frac{1}{2}M\lambda^A\lambda_A - (\frac{1}{6}a_{ijk}A_iA_jA_k + \frac{1}{2}b_{ij}A_iA_j + t_iA_i + \frac{1}{2}c_{ijk}A_i^*A_jA_k + c.c.) \\
&\quad -m_{ij}^2A_i^*A_j
\end{aligned}$$

Note that to be viable S̶U̶S̶Y̶ should to predict (universal) structures for the many soft-term parameters involved. Non-diagonal parameters tend to lead to flavor changing neutral currents (FCNC) or CP-violation in violation of measurement and should be avoided.

4.9 The hierarchy problem

Take a scalar particle, say the Higgs h. If we calculate loop-corrections to its mass in self-energy diagrams like the ones shown in Fig. 4.1, where f is a fermion and s some other scalar, they diverge, meaning they are infinite. This then needs what is called regularization in field theory in order to yield a finite answer. There are different ways of achiving this. Since we know that the SM is an incomplete theory, at least when we go up to Planck scale energies where we need an unknown quantum theory of gravity, we can introduce a cut-off regularization limiting the integral in the loop-correction to energies below a scale Λ_{UV}. Then the loop-correction to the Higgs mass is, at leading order in Λ_{UV},

$$\Delta m_h^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{UV}^2 + \frac{\lambda_s}{16\pi^2}\Lambda_{UV}^2 + \ldots \tag{4.24}$$

where λ_f and λ_s are the couplings of f and s to the Higgs, respectively, and Λ_{UV} is the high energy cut-off scale, suggestively the Planck scale, $\Lambda_{UV} = M_P = 2.4 \times 10^{18}$ GeV. Now, in order to keep $m_h \sim 125$ GeV as measured there must then be a crazy cancellation of 10^{16} times larger terms. This is known as the **hierarchy problem**.[21]

Enter supersymmetr to the rescue: with unbroken supersymmetry we find that we automatically have $|\lambda_f|^2 = \lambda_s$ and exactly twice as many scalar as fermion degrees of freedom running around in loops. This provides a magic cancellation of the quadratic divergence in Eq. (4.24). To see that this relation between the couplings holds, remember that

---

[20]We have omitted terms that have the form $-\frac{1}{2}m_{ij}\psi_i\psi_j$, because these can be absorbed by a redefinition of the superpotential.

[21]What about choosing dimensional regularization instead where there is no cut-off scale? That could in principle work, however, as soon as you introduce *any* new particle (significantly) heavier than the Higgs this results in a quadratic correction with the new particle mass, meaning that we cannot complete the SM at a higher scale without reintroducing the problem!
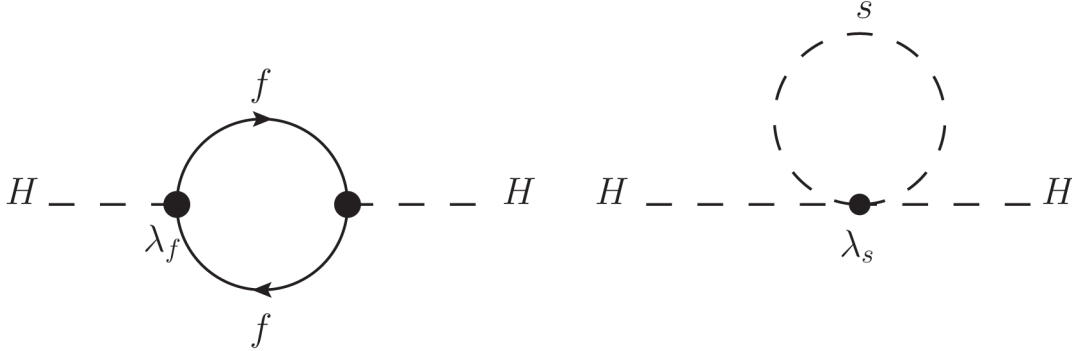
Figure 4.1: One loop contributions to the Higgs mass from a fermion (left) and scalar (right) loop.

$W \sim \lambda_{ijk}\Phi_i\Phi_j\Phi_k$ gives Lagrangian terms of the form $\lambda_{ijk}\psi_i\psi_j A_k$, and from the scalar potential we have terms of the form

$$V(A, A^*) \sim \left|\frac{\partial W}{\partial A_i}\right|^2 = |\lambda_{ijk}|^2 A_j^* A_k^* A_j A_k. \tag{4.25}$$

When the scalar field A_k is the Higgs field, the fermion is represented by $\psi_i = \psi_j$ and the second scalar by A_j, these two terms are responsible for the two types of vertices in Fig. 4.1 with $\lambda_f = \lambda_{ijk}$ and $\lambda_s = |\lambda_{ijk}|^2$. Note that the argument above applies to *any* scalar in the theory.

Now, we have unfortunately already broken supersymmetry, so what happens in ~~SUSY~~? This is the reason for restricting ourselves to soft supersymemtry breaking terms in the previous section. This guarantees that we end up with contributions to the Higgs mass of at most

$$\Delta m_h^2 = -\frac{\lambda_s}{16\pi^2}m_s^2 \ln\frac{\Lambda_{UV}^2}{m_s^2} + \dots, \tag{4.26}$$

at the leading order in Λ_{UV}, where m_s is the mass scale of the soft term. This is the most important argument in favour of supersymmetry existing at low energy scales where we can detect it, because m_s can not be too large if we want the above corrections to be small. This is called the **little hierarchy problem** and means that we want $m_s \sim \mathcal{O}(1 \text{ TeV})$ in order to keep cancellations reasonable.

4.10 The non-renormalization theorem

With our generic supersymmetric Lagrangian in Eq. (4.12) we should really ask ourselves whether we can regularize the theory, *i.e.* is there a finite number of renormalisation constants/counter terms to make all measurable predictions finite? And if so, what are they?

You may not be so surprised that the answer is yes, and indeed we have already used one of the restrictions this gives on the possible terms in our superpotential construction. Furthermore, we can prove the following theorem with a funny name...

> **Theorem**: **Non-renormalisation theorem** (Grisaru, Roach and Siegel, 1979 [9])
> All higher order contributions to the effective supersymmetric action S_{eff} can be written:
>
> $$S_{\text{eff}} = \sum_n \int d^4x_i...d^4x_n d^4\theta \, F_1(x_1, \bar{\theta}, \theta) \times ... \times F_n(x_1, \bar{\theta}, \theta) \times G(x_1, ..., x_n), \quad (4.27)$$
>
> where F_i are products of the external superfields and their covariant derivatives, and G is a supersymmetry invariant function.

So, why is the name funny? Well, mainly because it is not about not being able to renormalize the theory, but about about not *needing* to renormalize certain parts of it. The theorem has two important consequences:[22]

1. The couplings of the superpotential do not need separate normalization.

2. There is zero vacuum energy in global unbroken SUSY. In other words, $\Lambda = 0$ in general relativity.

3. Quantum corrections cannot (perturbatively) break supersymmetry.

Let us try to argue how these consequences come about. From the non-renormalization theorem we know that there are no counter terms needed for superpotential terms, because superpotential terms have lower θ integration than found in all the possible higher order contributions in the non-renormalisation theorem. This means that we can relate the bare fields Φ_0 and couplings g_0, m_0 and λ_0 to the renormalized fields Φ and couplings g, m and λ, by

$$g_0\Phi_0 = g\Phi, \quad (4.28)$$
$$m_0\Phi_0\Phi_0 = m\Phi\Phi, \quad (4.29)$$
$$\lambda_0\Phi_0\Phi_0\Phi_0 = \lambda\Phi\Phi\Phi. \quad (4.30)$$

If we let scalar superfields be renormalized by the **counterterm** Z, $\Phi_0 = Z^{1/2}\Phi$, vector superfields by Z_V, $V_0 = Z_V^{1/2}V$, coupling constant g by Z_g, $g_0 = Z_g g$, m by Z_m, $m_0 = Z_m m$, and λ by Z_λ, $\lambda_0 = Z_\lambda\lambda$, then

$$Z_g Z^{1/2} = 1 \quad (4.31)$$
$$Z_m Z^{1/2} Z^{1/2} = 1 \quad (4.32)$$
$$Z_\lambda Z^{1/2} Z^{1/2} Z^{1/2} = 1 \quad (4.33)$$

This set of equations can be solved for Z_g, Z_m and Z_λ in terms of $Z^{1/2}$ so no separate renormalization except for the superfields Φ and V is needed.

---

[22]The theorem is for unbroken supersymmetry.

The second consequence comes about because vaccum diagrams have no external fields. This means that the integration $\int d^4\theta$ in S_{eff} gives zero for the contribution from these diagrams. The same argument leads to $V(A, A^*) = 0$ after quantum corrections.

In practice the regularisation of supersymmetric models is tricky. Using so-called DREG (dimensional regularisation) with modified minimal subtraction ($\overline{MS}$) fails because working in $d = 4 - \epsilon$ dimensions violates the supersymmetry in the Lagrangian. In practice DRED (dimensional reduction) with $\overline{DR}$ is used, where all the algebra is done in four dimensions, but integrals are done in $d = 4 - \epsilon$ dimensions. However, this leads to its own problems with potential ambiguities in higher loops.

4.11 Renormalisation group equations

Renormalisation, the removal of infinities from field theory predictions, introduces a fixed scale μ at which the parameters of the Lagrangian, the couplings, are defined. For example, the charge of the electron is not simply the bare charge e, but a charge at a given energy scale μ, $e(\mu)$, which is the scale at which the theory describes the electron, and which we can measure in an experiment at that scale. Scattering an electron at very high energy will require a different value of $e(\mu)$ than at a low energy. This is an experimentally well verified fact.[23]

However, since μ is not an observable *per se* but in principle a choice of how to write down the theory (at which energy to write down the Lagrangian), the action should be invariant under a change of μ, which is expressed as:

$$\mu \frac{d}{d\mu} S(Z\Phi, \lambda, \mu) = 0, \tag{4.34}$$

where λ are the couplings of the theory and Φ represents the (super)fields that have been renormalised.[24] This equation can be re-written in terms of partial derivatives

$$\left(\mu \frac{\partial}{\partial\mu} + \mu \frac{\partial\lambda}{\partial\mu} \frac{\partial}{\partial\lambda} \right) S(Z\Phi, \lambda, \mu) = 0, \tag{4.35}$$

which is the **renormalisation group equation** (RGE).

We can look at the behavior of a Lagrangian parameter λ as a function of the energy scale μ away from the value where it was defined, often denoted μ_0. This is controlled by the β-**function**:

$$\beta_\lambda \equiv \mu \frac{\partial\lambda}{\partial\mu}. \tag{4.36}$$

These β-functions can be found from the counterterm Z. As an example, take a **gauge coupling constant** g_0 defined (taken from measurement) at some scale μ_0. At a different scale μ, g_0 is given by (in $d = 4 - \epsilon$ dimensions):[25]

$$g_0 = Zg\mu^{-\epsilon/2}$$

---

[23]It is also impossible to avoid if we accept that the electron is a point particle. Since the potential has the form $V(r) \propto e/r$ an infinte energy would appear unless we somehow were to modify the charge at high energies, or equivalently short distances.

[24]In the previous section we showed that we did not need to renormalise the coupling constants of the superpotential.

[25]The factor $\mu^{-\epsilon/2}$ is there to ensure that the scale of g is correct, see the exercise below.

Then, differentiating both sides with respect to μ,

$$
\begin{aligned}
0 &= \frac{\partial Z}{\partial \mu} g \mu^{-\epsilon/2} + Z \frac{\partial g}{\partial \mu} \mu^{-\epsilon/2} - \frac{\epsilon}{2} Z g \mu^{-\epsilon/2 - 1} \\
\mu \frac{\partial g}{\partial \mu} &= \frac{\epsilon}{2} g - \frac{g \mu}{Z} \frac{\partial Z}{\partial \mu} \\
\mu \frac{\partial g}{\partial \mu} &= \frac{\epsilon}{2} g - g \mu \frac{\partial}{\partial \mu} \ln Z,
\end{aligned}
$$

and taking the limit $\epsilon \to 0$:

$$
\beta_g = \mu \frac{\partial g}{\partial \mu} = -g \gamma_g,
$$

where we have defined the anomalous dimension of g

$$
\gamma_g = \mu \frac{\partial}{\partial \mu} \ln Z. \tag{4.37}
$$

It is often practical to rewrite $\beta_g = \frac{\partial g}{\partial t}$ with $t = \ln \mu$ so that $\mu \frac{\partial}{\partial \mu} = \frac{\partial}{\partial t}$.

Z can now be calculated to the required loop-order to find the β-function to that order and in turn the running of the coupling constant with μ. By evaluating one-loop super graphs we can find that for our particular example

$$
\gamma_g \big|_{1-\text{loop}} = \frac{1}{16\pi^2} g^2 \left(\sum_R T(R) - 3C(A) \right), \tag{4.38}
$$

where the sum is over all superfields that transform under a representation R of the gauge group and $C(A)$ is the Casimir invariant of the adjoint representation A of R. This expression is particularly important since it will later lead us to the concept of gauge coupling unification. Notice both that the running of the couplings with scale μ is very slow because the β-function is a logarithmic function of μ and that the anomolous dimension may be negative for some gauge groups.

Exercise: For fun, and ten points, prove the scale factor in $g_0 = Zg\mu^{-\epsilon/2}$. *Hint: what are the dimensions of stuff in the Lagrangian in $d = 4 - \epsilon$ dimensions?*

4.12   Vacuum energy

We saw in the Section 4.10 that a globaly supersymmetric theory has $\Lambda = 0$. This is to be compared to the measured value of the dark energy density, which can be interpreted as vacuum energy and is $\Lambda_{DE} \sim 10^{-3}$ eV, and the value in the SM which is $\Lambda \sim M_P \simeq 10^{18}$ GeV.[26] Clearly models with supersymmetry are doing a bit better than the SM in predicting this. Now, what about ~~SUSY~~?

---

[26]The origin of this is just the same as the quadratic divergence for the Higgs mass. It is the same type of diagrams contributing, only without external legs.

The scale of the contribution has to be the mass scale of the supersymmetric particles, so with $m_{SUSY} \geq 1\,\text{TeV}$ we have $m_{SUSY}/\Lambda_{DE} \geq 10^{15}$ which is twice as good as $M_P/\Lambda_{DE} = 10^{30}$ but still a bit off the measured value. This problem is the **hierachy problem for vacuum energy**.

However, in supergravity something interesting happens. Introducing a local supersymmetry the scalar potential is not simply given by the superpotential derivatives in (4.17), but instead is (ignoring the effects of gauge fields)

$$V(A, A^*) = e^{K/M_P} \left[K_{ij}(D_i W)(D_j W^*) - \frac{3}{M_P^2} |W|^2 \right], \qquad (4.39)$$

where $K(A, A^*)$ is the so-called **Kähler potential**, $K_{ij} = \partial_i \partial_j K$ is the **Kähler metric** (the derivatives are with respect to the scalar fields) and D_i the **Kähler derivative** $D_i = \partial_i + \frac{1}{M_P^2}(\partial_i K)$. In the $M_P \to \infty$ limit, the low energy limit, we see that we recover the flat space result of Eq. (4.17). What is important to notice is that there is now a second *negative* term in the potential that can in principle cancel the ~~SUSY~~ contribution, however, this will come at the price of fantastic fine-tuning unless some mechanism can be found where this is natural.

Chapter 5

The Minimal Supersymmetric Standard Model (MSSM)

The Minimal Supersymmetric Standard Model (MSSM) is a minimal model in the sense that it has the smallest field (and gauge) content consistent with the known SM fields. We will now construct this model on the basis of the previous chapters, and look at some of its consequences.

5.1 MSSM field content

Previously we learnt that each (left-handed) scalar superfield S has a (left-handed) Weyl spinor ψ_A and a complex scalar $\tilde{s}$ since they are a $j = 0$ representation of the superalgebra.[1] Given an application of the equations of motion these have two fermionic and two bosonic degree of freedom remaining each (the auxiliary field has been eliminated and with it two fermionic d.o.f.).

In order to construct a Dirac fermion, which are plentiful in the SM, we need a right-handed Weyl spinor as well. We can aquire the needed right-handed Weyl spinor from the $\bar{T}^\dagger$ of a different scalar superfield $\bar{T}$ with the right-handed Weyl spinor $\bar{\varphi}_{\dot{A}}$.[2] With these four fermionic d.o.f. we can construct *two* Dirac fermions, a particle–anti-particle pair, and four scalars, two particle–anti-particle pairs.

We use these two superfield ingredients to construct all the known fermions:

- To get the SM leptons we introduce the superfields l_i and $\bar{E}_i$ for the charged leptons (i is the generation index) and ν_i for the neutrinos, where we form $SU(2)_L$ doublet vectors $L_i = (\nu_i, l_i)$. We do not introduce $\bar{N}_i$.[3] These would contain right-handed neutrino spinors needed for massive Dirac neutrinos, but are omitted as they do not couple to anything, being SM singlets.[4] This is a convention (MSSM is older than neutrino mass),

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[1]With all posssible appologies, we have now changed notation for these fields to what is conventional in phenomenology (as opposed to pure theory) and we will try to use the tilde notation for the scalar component fields, while the superfields are denoted by latin letters.

[2]The bar here is used to (not) confuse us, it is part of the name of the superfields and does not denote any hermitian or complex conjugate.

[3]The anti-neutrino contained in the superfield $\nu_i^\dagger$ is right-handed consistent with experiment.

[4]They can't be colour-charged, they are right-handed singlets under $SU(2)_L$ thus they have zero weak isospin, but since they should also have zero electric charge the hypercharge must also be zero.

and including $\bar{N}_i$ fields has some interesting consequences.[5]

- For quarks the situation is similar. Up-type and down-type quarks get the superfields u_i, $\bar{U}_i$ and d_i, $\bar{D}_i$, forming the $SU(2)_L$ doublets $Q_i = (u_i, d_i)$.[6]

Additionally we need vector superfields, which after the e.o.m. contain a *massless* vector boson with two scalar d.o.f. and two Weyl-spinors, one of each handedness λ and $\bar{\lambda}$, with two fermionic degrees of freedom. Together these form a $j = \frac{1}{2}$ representation of the superalgebra. If the vector superfield is neutral, the fermions can form a Majorana fermion, if not they can be combined with the Weyl-spinors from other fields to form Dirac fermions.

Looking at the construction $V \equiv q t^a V^a$ in the supersymmetric Lagrangian we see that, as expected, we need one superfield V^a per generator t^a of the algebra, giving the normal $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ vector bosons. We call these superfields C^a, W^a and B^0.[7] In order to be really confusing, we use the following symbols for the fermions constructed from the respective Weyl-spinors: $\tilde{g}$, $\tilde{W}^0$ and $\tilde{B}^0$. The tilde here is supposed to tells us that hey are supersymmetric partners (often just called **sparticles**) of the known SM particles.

We also need Higgs superfields. Now life gets interesting. The usual Higgs $SU(2)_L$ doublet sclar field H in the SM cannot give mass to all fermions because it relies on the $H^C \equiv -i(H^\dagger \sigma_2)^T$ construction to give masses to up-type quarks (and possibly neutrinos). The superfield version of this cannot appear in the superpotential because it would mix left- and right-handed superfields. The minimal Higgs content we can get away with are two Higgs superfield $SU(2)_L$ doublets, which we will call H_u and H_d, indexing the quarks they give mass to.[8] These must have (more on that in a little bit) weak hypercharge $y = \pm 1$ for H_u and H_d respectively, so that we have the doublets:

$$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}, \quad H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}. \tag{5.1}$$

5.2   The kinetic terms

It is now straight forward to write down the kinetic terms of the MSSM Lagrangian giving matter-gauge interaction terms

$$\begin{aligned}
\mathcal{L}_{kin} &= L_i^\dagger e^{\frac{1}{2}g\sigma W - \frac{1}{2}g'B} L_i + Q_i^\dagger e^{\frac{1}{2}g_s\lambda C + \frac{1}{2}g\sigma W + \frac{1}{3}\cdot\frac{1}{2}g'B} Q_i \\
&+ \bar{U}_i^\dagger e^{\frac{1}{2}g_s\lambda C - \frac{4}{3}\cdot\frac{1}{2}g'B} \bar{U}_i + \bar{D}_i^\dagger e^{\frac{1}{2}g_s\lambda C + \frac{2}{3}\cdot\frac{1}{2}g'B} \bar{D}_i \\
&+ \bar{E}_i^\dagger e^{2\frac{1}{2}g'B} \bar{E}_i + H_u^\dagger e^{\frac{1}{2}g\sigma W + \frac{1}{2}g'B} H_u + H_d^\dagger e^{\frac{1}{2}g\sigma W - \frac{1}{2}g'B} H_d,
\end{aligned} \tag{5.2}$$

where g', g and g_s are the couplings of $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$. As a convention we assign the charge under $U(1)$, hypercharge, in units of $\frac{1}{2}g'$. All non-singlets of $SU(2)_L$ and $SU(3)_C$ have the same charge, the factor $\frac{1}{2}$ here is used to get by without accumulation of numerical factors since the algebras for the Pauli and Gell-Mann matrices are:

$$\left[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j\right] = i\epsilon_{ijk}\frac{1}{2}\sigma_k,$$

---

[5]Note that component fields in the same superfield must have the same charge under all the gauge groups, *i.e.* the scalar partner of the electron has electric charge $-e$, so it cannot be a neutrino.

[6]Here we should really also include a color index a such that u_i^a is a component in a $SU(3)_C$ vector. We omit these for simplicity.

[7]And there we have another W.

[8]In some further insanity some authors prefer H_1 and H_2 so that you have no idea which is which.

and

$$\left[\frac{1}{2}\lambda_i, \frac{1}{2}\lambda_j\right] = i f_{ijk}\frac{1}{2}\lambda_k.$$

These conventions lead to the SM gauge transformations for fermion component fields and the familiar relations after electroweak symmetry breaking,[9] $Q = \frac{y}{2} + T_3$, where Q is the unit of electric charge, y is hypercharge and T_3 is weak charge, and $e = g \sin\theta_W = g' \cos\theta_W$.

We mentioned earlier that the two Higgs superfields have opposite hypercharge. This is needed for so-called **anomaly cancellation** in the MSSM. Gauge anomaly is the possibility that at loop level contributions to processes such as in Fig. 5.1 break gauge invariance and ruins the predictability of the theory. This miraculously does not happen in the SM becuase it has the field content it has, so that all gauge anomalies cancel (we don't know of a deeper reason). If we have *one* Higgs doublet this does not happer for the MSSM. With two Higgs doublets, with opposite hypercharge, it does.
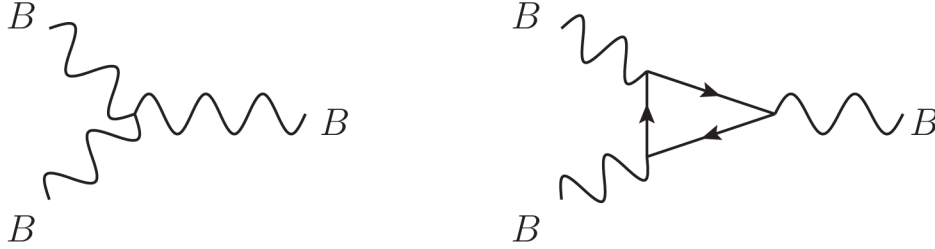

Figure 5.1: Possible three gauge boson B couplings a one-loop fermion contribution.

5.3 Gauge terms

The pure gauge terms with supersymmetric field strengths are also fairly easy to write down:

$$\mathcal{L}_V = \frac{1}{2}\text{Tr}\{W^A W_A\}\theta\theta + \frac{1}{2}\text{Tr}\{C^A C_A\}\theta\theta + \frac{1}{4}B^A B_A\theta\theta + \text{h.c.} \qquad (5.3)$$

where we have used

$$T(R)_L = \text{Tr}\left[\frac{1}{2}\sigma^1 \cdot \frac{1}{2}\sigma^1\right] = \frac{1}{2},$$

and

$$T(R)_C = \text{Tr}\left[\frac{1}{2}\lambda^1 \cdot \frac{1}{2}\lambda^1\right] = \frac{1}{2},$$

in the normalization of the terms, and where the field strengths are given as:

$$W_A = -\frac{1}{4}\bar{D}\bar{D}e^{-W}D_A e^W, \qquad W = \frac{1}{2}g\sigma^a W^a, \qquad (5.4)$$

$$C_A = -\frac{1}{4}\bar{D}\bar{D}e^{-C}D_A e^C, \qquad C = \frac{1}{2}g_s\lambda^a C^a, \qquad (5.5)$$

$$B_A = -\frac{1}{4}\bar{D}\bar{D}D_A B, \qquad B = \frac{1}{2}g' B^0. \qquad (5.6)$$

---

[9]Getting ahead of ourselves a little here.

5.4   The MSSM superpotential

With the same gauge structure as in the SM in place we are ready to write down all possible terms in the superpotential. First, we notice that there can be no **tadpole terms** (terms with only one superfield), since there are no superfields that are singlets (zero charge) under all SM gauge groups. The only alternative would be right-handed neutrino superfields $\bar{N}_i$.

We have seen that possible mass terms must fulfill $m_{ij}U_{ir}U_{js} = m_{rs}$ to preserve gauge invariance. For the abelian gauge group $U(1)_Y$ this reduces to $Y_i + Y_j = 0$, which is easier to check so this is where we start. In Table 5.1 we see that the only possible contributions are particle–anti-particle combinations such as $l_{iL}\bar{l}_{iR}$, but these come from superfields with different handedness and cannot be used together.

Superfield	L_i	$\bar{E}_i^\dagger$	Q_i	$\bar{U}_i^\dagger$	$\bar{D}_i^\dagger$
Particle	$\nu_{iL},\, l_{iL}$	l_{iR}	u_{iL}, d_{iL}	u_{iR}	d_{iR}
Hypercharge	-1	-2	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$
Superfield	$L_i^\dagger$	$\bar{E}_i$	$Q_i^\dagger$	$\bar{U}_i$	$\bar{D}_i$
Anti-particle	$\bar{\nu}_{iR},\, \bar{l}_{iR}$	$\bar{l}_{iL}$	$\bar{u}_{iR}, \bar{d}_{iR}$	$\bar{u}_{iL}$	$\bar{d}_{iL}$
Hypercharge	1	2	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$

Table 5.1: MSSM superfields with SM fermion content and their hypercharge.

The exception is for the two Higgs superfields that have opposite hypercharge. In order to also be invariant under $SU(2)_L$ we have to write this superpotential term as

$$\mathcal{L}_{\text{mass}} = \mu H_u^T i\sigma^2 H_d, \tag{5.7}$$

where μ is the Lagrangian mass parameter.[10] This is invariant under $SU(2)_L$ because, with the gauge transformations $H_d \rightarrow e^{ig\frac{1}{2}\sigma^k W^k} H_d$ and $H_u^T \rightarrow H_u^T e^{ig\frac{1}{2}\sigma^{kT} W^k}$, we get

$$\begin{aligned}
H_u^T i\sigma_2 H_d &\rightarrow H_u^T e^{ig\frac{1}{2}\sigma^{kT} W^k} i\sigma_2 e^{ig\frac{1}{2}\sigma^k W^k} H_d \\
&= H_u^T i\sigma^2 e^{-i\frac{1}{2}g\sigma^k W^k} e^{i\frac{1}{2}g\sigma^k W^k} H_d = H_u^T i\sigma^2 H_d,
\end{aligned}$$

since $\sigma^{kT}\sigma^2 = -\sigma^2\sigma^k$. Usually we ignore the $SU(2)_L$ specific structure and write terms like this as $\mu H_u H_d$, confusing the hell out of anyone that is not used to this convention since we really do mean Eq. (5.7). Notice that if we write (5.7) in terms of component fields we get

$$H_u^T i\sigma^2 H_d = H_u^+ H_d^- - H_u^0 H_d^0,$$

which we should have been able to guess because the Lagrangian must also conserve electric charge.

If you have paid very close attention to the argument above you may have noticed that there is one more possibility, namely

$$\mu_i' L_i H_u \equiv \mu_i' L_i^T i\sigma^2 H_u = \mu_i'(\nu_i H_u^0 - l_i H_u^+),$$

where μ' is some other mass parameter in the superpotential. This is clearly an allowable term (and we will return to it below), however, it also raises a very interesting question:

---

[10]Must not be confused with the RGE scale!

Could we have $L_i \equiv H_d$? Could the lepton superfields L_i play the rôle of Higgs superfields, thus reducing the field content needed to describe the SM particles in a supersymmetric theory? While not immediately forbidden, this suggestions unfortunately leads to problems with anomaly cancelation, processes with large lepton flavor violation (LFV) and much too massive neutrinos, and has been abandoned.

We have now found all possible mass terms in the superpotential. What about the Yukawa terms? The hypercharge requirement is $Y_i + Y_j + Y_k = 0$. From our table of hypercharges only the following terms are viable:

$$L_i L_j \bar{E}_k \ , \qquad L_i H_d \bar{E}_j \ , \qquad L_i Q_j \bar{D}_k \ , \qquad Q_i H_u \bar{U}_j \ , \qquad \bar{U}_i \bar{D}_j \bar{D}_k \qquad \text{and} \qquad Q_i H_d \bar{D}_i.$$

For all these terms we can simultaneously keep $SU(2)_L$ invariance with the $i\sigma^2$ construction implicitly inserted between any superfield doublets.

For $SU(3)_C$ to be conserved, we need to have colour singlets. Some of these terms are colour singlets by construction since they do not contain any coloured fields. The terms with two quark superfields contain left-handed Weyl spinors for quarks and anti-quarks, which are $SU(3)_C$ singlets if the superfields come in colour–anti-colour pairs. In representation language they are in the **3** and $\bar{\mathbf{3}}$ representations of $SU(3)_C$. Written with all indices explicit we have e.g. $L_i Q_j \bar{D}_k = L_i Q_j^\alpha i\sigma^2 \bar{D}_{k\alpha}$, where α is the colour index. The final term $\bar{U}_i \bar{D}_j \bar{D}_k$ is a colour singlet once we demand that it is totally anti-symmetric in the colour indices: $\bar{U}_i \bar{D}_j \bar{D}_k \equiv \epsilon^{\alpha\beta\gamma} \bar{U}_{i\alpha} \bar{D}_{j\beta} \bar{D}_{k\gamma}$.

Our complete superpotential is then:

$$\begin{aligned} W \quad = \quad & \mu H_u H_d + \mu_i' L_i H_u + y_{ij}^e L_i H_d E_j + y_{ij}^u Q_i H_u \bar{U}_j + y_{ij}^d Q_i H_d \bar{D}_j \\ & + \lambda_{ijk} L_i L_j \bar{E}_k + \lambda'_{ijk} L_i Q_j \bar{D}_k + \lambda''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k, \end{aligned} \tag{5.8}$$

where we have named and indexed the couplings in a natural way.[11]

> **Exercise**: Using the explicit form of the $SU(3)_C$ transformations with the Gell-Mann matrices, show that with our definition of the superpotential term $\bar{U}_i \bar{D}_j \bar{D}_k$ this is invariant under $SU(3)_C$.

5.5 R-parity

The superpotential terms LH_u, LLE and $LQ\bar{D}$ that we have written down all violate lepton number conservation, and $\bar{U}\bar{D}\bar{D}$ violates baryon number conservation. Allowing such terms leads to, among other phenomenological problems, processes like proton decay $p \to e^+ \pi^0$ as shown in Fig. 5.2.

We can estimate the resulting proton life-time by noting that the scalar particle (a strange squark $\tilde{s}$) creates an effective Lagrangian term $\lambda \bar{u}\bar{d}eu$ with coupling

$$\lambda = \frac{\lambda'_{112} \lambda''_{112}}{m_{\tilde{s}}^2}, \tag{5.9}$$

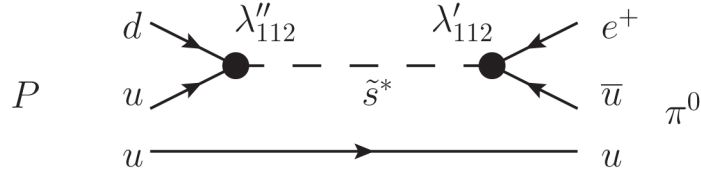[11] For some peculiar opinion of what is natural.

Figure 5.2: Feynman diagram for proton decay with RPV couplings.

where the sparticle mass $m_{\tilde{s}}$ comes from the scalar propagator in the diagram. The resulting matrix element for the process must then be proportional to $|\lambda|^2$. Since the mass scale involved in the problem is the proton mass m_p the phase space integration part of a calculation of the proton decay width must be of the order of m_p^5. We then have

$$\Gamma_{p \to e^+ \pi^0} \sim |\lambda|^2 m_p^5 = \frac{|\lambda'_{112} \lambda''_{112}|^2}{m_{\tilde{s}}^4} m_p^5. \tag{5.10}$$

The measured lower limit on the lifetime from watching a lot of protons not decay is $\tau_{p \to e^+ \pi^0} > 1.6 \cdot 10^{33}$ y or $\tau_{p \to e^+ \pi^0} > \pi \cdot 10^7$ s/y $\times 1.6 \cdot 10^{33}$ y $= 5.0 \cdot 10^{40}$ s, which gives $\Gamma_{p \to e^+ \pi^0} < 1.3 \cdot 10^{-65}$ GeV, so that with we have the following very strict limit on the combination of two couplings

$$|\lambda'_{112} \lambda''_{112}| < 3.6 \cdot 10^{-27} \sqrt{\frac{m_{\tilde{s}}}{1 \text{ TeV}}}. \tag{5.11}$$

To avoid all such couplings Fayet (1975) [10] introduced the conservation of R-partity.

> **Definition**: **R-parity** is a multiplicatively conserved quantum number given by
>
> $$R = (-1)^{2s+3B+L}$$
>
> where s is a particle's spin, B its baryon number and L its lepton number.

For all SM particles $R = 1$, while the superpartners all have $R = -1$. One usually *defines* the MSSM as conserving R-parity. The consequence of this somewhat *ad hoc* definition is that in all interactions supersymmetric particles are only created or annihilated in pairs. This leads to the following very important phenomenological consequences:

1. The lightest supersymmetric particle (LSP) is absolutely stable.

2. Every other sparticle must decay down to the LSP (possibly in multiple steps).

3. Sparticles will always be produced in pairs in collider experiments.

For the MSSM this excludes the terms LH_u, $LL\bar{E}$, $LQ\bar{D}$ and $\bar{U}\bar{D}\bar{D}$ from the superpotential.

5.6 SUSY breaking terms

We can use our previous arguments on gauge invariance that we used when discussing the superpotential on the soft-breaking terms to determine which terms are allowed. Terms

$$-\frac{1}{4T(R)}M\theta\theta\bar{\theta}\bar{\theta}\text{Tr}\{W^A W_A\},$$

are allowed because they have the same gauge structure as the field strength terms. In component fields these are for the MSSM:

$$-\frac{1}{2}M_1\tilde{B}\tilde{B} - \frac{1}{2}M_2\tilde{W}^{iA}\tilde{W}^i_A - \frac{1}{2}M_3\tilde{g}^{aA}\tilde{g}^a_A + c.c$$

where the M_i are potentially complex-valued. This gives six new parameters. Terms

$$-\frac{1}{6}a_{ijk}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j\Phi_k,$$

are allowed when corresponding terms exist in the superpotential (are gauge invariant and not disallowed by R-parity). In component fields the allowed terms are

$$-a^e_{ij}\tilde{L}_i H_d \tilde{e}^*_{jR} - a^u_{ij}\tilde{Q}_i H_u \tilde{u}^*_{jR} - a^d_{ij}\tilde{Q}_i H_d \tilde{d}^*_{jR} + c.c.$$

where the H here refers to scalar parts of the Higgs superfields. The couplings a_{ij} are all potentially complex valued, so this gives us 54 new parameters. The terms

$$-\frac{1}{2}b_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j,$$

are only allowed for corresponding terms in the superpotential, *i.e.* $-bH_u H_d + c.c.$, where b is potentially complex valued, which gives us 2 new parameters.[12] Tadpole terms

$$-t_i\theta\theta\bar{\theta}\bar{\theta}\Phi_i,$$

are not allowed, as there are no tadpoles in the superpotential. Mass terms

$$-m^2_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi^\dagger_i\Phi_j,$$

are allowed because they have the same gauge structure as kinetic terms. In component fields they are:

$$-(m^L_{ij})^2\tilde{L}^\dagger_i\tilde{L}_j - (m^e_{ij})^2\tilde{e}^*_{iR}\tilde{e}_{jR} - (m^Q_{ij})^2\tilde{Q}^\dagger_i\tilde{Q}_j - (m^u_{ij})^2\tilde{u}^*_{iR}\tilde{u}_{jR} - (m^d_{ij})^2\tilde{d}^*_{iR}\tilde{d}_{jR}$$
$$-m^2_{H_u}H^\dagger_u H_u - m^2_{H_d}H^\dagger_d H_d \tag{5.12}$$

where the m^2_{ij} are complex valued, however, also hermetic. This gives rise to 47 new parameters. Despite being allowed the MSSM ignores the "maybe-soft" terms in Eq. (4.23).

In total, after using our freedom to choose our basis wisely in order to remove what freedom we can, the MSSM has 105 new parameters compared to the SM, 104 of these are soft-breaking terms and μ is the only new parameter in the superpotential.

---

[12]The coupling b is sometimes written $B\mu$ where B is a unitless constant that indicates how different the coupling is from the corresponding coupling in the superpotential.

5.7   Radiative EWSB

In the SM the vector bosons are given mass spontaneous by electroweak symmetry breaking (EWSB), which is induced by the shape of the scalar potential for a scalar field Φ:

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2, \tag{5.13}$$

with $\lambda > 0$ and $\mu^2 < 0$.[13] In supersymmetry we have the scalar potential

$$V(A, A^*) = \sum_i \left| \frac{\partial W}{\partial A_i} \right|^2 + \frac{1}{2} \sum_a g_a^2 (A^* T^a A)^2 > 0, \tag{5.14}$$

when we have extended Eq. (4.17) by including also gauge interactions and vector super-fields.[14] For the scalar Higgs component fields (not superfields!) this gives the MSSM potential

$$
\begin{aligned}
V(H_u, H_d) \;=\; & |\mu|^2 (|H_u^0|^2 + |H_u^+|^2 + |H_d^0|^2 + |H_d^-|^2) && \text{(from F-terms)} \\
& + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2 && \text{(from D-terms)} \\
& + \frac{1}{2}g^2 |H_u^+ H_d^{0*} + H_u^0 H_d^{-*}|^2 \\
& + m_{H_u}^2 (|H_u^0|^2 + |H_u^+|^2) + m_{H_d}^2 (|H_d^0|^2 + |H_d^-|^2) && \text{(from soft breaking terms)} \\
& + [b(H_u^+ H_d^- - H_u^0 H_d^0) + c.c] && (5.15)
\end{aligned}
$$

This potential has 8 d.o.f. from 4 complex scalar fields H_u^+, H_u^0, H_d^0 and H_d^-.

We now want to do as in the SM and break $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$ in order to give masses to gauge bosons and SM fermions.[15] To do this we need to show that (5.15) has: i) a minimum for finite, *i.e.* non-zero, field values, ii) that this minimum has a remaining $U(1)_{\text{em}}$ symmetry and iii) that the potential is bunded from below, which are the essential properties of Eq. (5.13). We restrict our analysis to tree level, ignoring loop effects on the potential.

We start by using our $SU(2)_L$ gauge freedom to rotate away any field value for H_u^+ at the minimum of the potential, so without loss of generality we can use $H_u^+ = 0$ in what follows. At the minimum we must have $\partial V / \partial H_u^+ = 0$, and by explicit differentiation of the potential one can show that $H_u^+ = 0$ then leads to $H_d^- = 0$. This is good since it guarantees our item ii), that $U(1)_{\text{em}}$ is a symmetry for the minimum of the potential, since the charged fields then have no vev. We are then left with the potential

$$
\begin{aligned}
V(H_u^0, H_d^0) \;=\; & (|\mu|^2 + m_{H_u}^2)|H_u^0|^2 + (|\mu|^2 + m_{H_d}^2)|H_d^0|^2 \\
& + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 - |H_d^0|^2)^2 - (bH_u^0 H_d^0 + c.c.) && (5.16)
\end{aligned}
$$

Since we can absorb a phase in H_u^0 or H_d^0 we can take b to be real and positive. This does not affect other terms because they are protected by absolute values. The minimum must also have $H_u^0 H_d^0$ real and positive, to get a as large as possible negative contribution from the b

---

[13]The **Mexican hat** or **wine bottle** potential, depending on preferences.

[14]The last term is due to the elimination of auxillary d-fields from vector superfields giving a contribution $d^a d^a = g_a^2 (A^* T^a A)^2$.

[15]The soft-terms are unable to provide masses to these particles because they deal mostly with scalar fields.

term. Thus the vevs $v_u = \langle H_u^0 \rangle$ and $v_d = \langle H_d^0 \rangle$ must have opposite phases. By the remaining $U(1)_Y$ symmetry, we can transform v_u and v_d so that they are real and have the same sign. For the potential to have a negative mass term, and thus fulfill point i) above, we must then have

$$b^2 > (|\mu|^2 + m_{H_u}^2)(|\mu|^2 + m_{H_d}^2). \tag{5.17}$$

Since the potential has ~~SUSY~~ we must also check that it is actually bounded from below, our point iii), which was guaranteed for the SUSY vacuum. For large $|H_u^0|$ or $|H_d^0|$ the quartic gauge term blows up to save the potential, except for $|H_u^0| = |H_d^0|$, the so-called d-flat directions. This means that we must also require

$$2b < 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2. \tag{5.18}$$

Negative values of $m_{H_u}^2$ (or $m_{H_d}^2$) help satisfy (5.17) and (5.18), but they do not guarantee EWSB. If we assume that $m_{H_d} = m_{H_u}$ at some high scale (GUT) then (5.17) and (5.18) cannot be simultaneously be satisfied *at that scale*. However, to 1-loop the RGE running of these mass parameters is:

$$16\pi^2 \beta_{m_{H_u}^2} \equiv 16\pi^2 \frac{dm_{H_u}^2}{dt} = 6|y_t|^2 (m_{H_u}^2 + m_{Q_3}^2 + m_{u_3}^2) + ...$$

$$16\pi^2 \beta_{m_{H_d}^2} \equiv 16\pi^2 \frac{dm_{H_d}^2}{dt} = 6|y_b|^2 (m_{H_d}^2 + m_{Q_3}^2 + m_{u_3}^2) + ...$$

where y_t and y_b are the top and bottom quark Yukawa couplings, and $m_{Q_3} = m_{33}^Q$, $m_{u_3} = m_{33}^u$, $m_{d_3} = m_{33}^d$ in our previous notation. Because $y_t \gg y_b$, m_{H_u} runs down much faster than m_{H_d} as we go to the electroweak scale, and may become negative, see Fig. 5.3. It is this property that is termed **radiative EWSB** (REWSB). Thus, in the MSSM with soft terms there is an explanation why EWSB happens, it is not put in by hand in the potential as it is in the SM!

To get the familiar vector boson masses, we need to satisfy the electroweak constraint:

$$v_u^2 + v_d^2 \equiv v^2 = \frac{2m_Z^2}{g^2 + g'^2} \approx (174 \text{ GeV})^2,$$

which comes from experiment. Thus we have one free parameter coming from the Higgs vevs. We can write this as

$$\tan\beta \equiv \frac{v_u}{v_d},$$

where by convention $0 < \beta < \pi/2$. Using the conditions $\partial V / \partial H_u^0 = \partial V / \partial H_d^0 = 0$ for the minimum, b and $|\mu|$ can be eliminated as free parameters from the model, however, not the sign of μ. Alternatively, we can choose to eliminate $m_{H_u}^2$ and $m_{H_d}^2$. You can look at this as giving away the freedom of these parameters to the vevs, and then fixing one vev by the electroweak constraint, and using $\tan\beta$ for the other.

Let us make a little remark here on the parameter μ. We have what is called the μ **problem**. The soft terms all get their scale from some common mechanism at some common high energy scale, it is assumed, however, μ is a mass term in the superpotential (the only one) and could *a priori* take *any* value, even M_P. Why is μ then of the order of the soft terms allowing us to achieve REWSB?[16]

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[16]This problem can be solved in extensions of the MSSM such as the Next-to-Minimal Supersymmetric Standard Model (NMSSM).
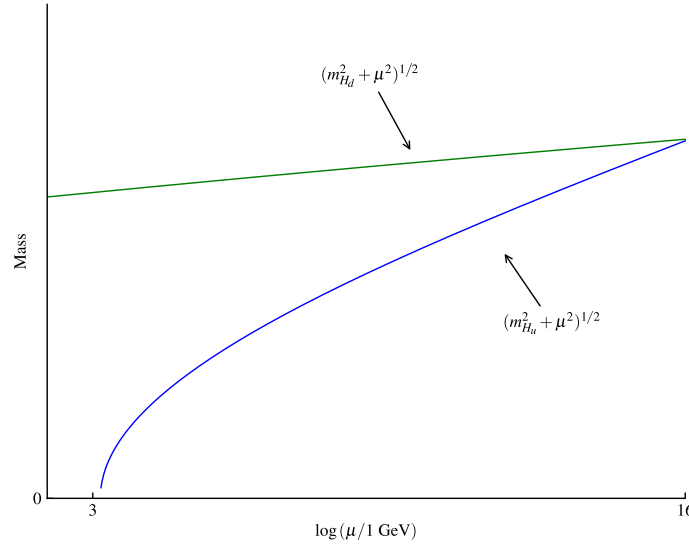
Figure 5.3: Sketch of the RGE running of the two soft Higgs mass parameters $m^2_{H_u}$ and $m^2_{H_d}$ as a function of the energy scale

Exercise: Show how you can eliminate the parameters $|\mu|$ and b by using the properties of the minimum of the potential in Eq. (5.16).

5.8   Higgs boson properties

Of the 8 d.o.f. in the scalar potential for the Higgs component fields three are Goldstone bosons that get eaten by Z and $W^\pm$ to give masses. The remaining 5 d.o.f. form two neutral scalars h, H, two charged scalars $H^\pm$ and one neutral pseudo-scalar (CP-odd) A.[17] At tree level one can show that these have the masses:

$$m^2_A = \frac{2b}{\sin 2\beta} = 2|\mu|^2 + m^2_{H_u} + m^2_{H_d}, \tag{5.19}$$

$$m^2_{h,H} = \frac{1}{2}\left(m^2_A + m^2_Z \mp \sqrt{(m^2_A - m^2_Z)^2 + 4m^2_Z m^2_A \sin^2 2\beta}\right), \tag{5.20}$$

$$m^2_{H^\pm} = m^2_A + m^2_W. \tag{5.21}$$

As a consequence m_A and $\tan\beta$ can be used to parametrize the Higgs sector (at tree level), and H, $H^\pm$ and A are in principle unbounded in mass since they grow as $b/\sin 2\beta$. However, at tree level the lightest Higgs boson is restricted to

$$m_h < m_Z|\cos 2\beta|. \tag{5.22}$$

---

[17]In addition to the scalars, the Higgs supermultiplets contain four fermions, $\tilde{H}^0_u$, $\tilde{H}^0_d$, $\tilde{H}^+_u$ and $\tilde{H}^-_d$ (higgsinos). These will mix with the fermion partners of the gauge bosons (gauginos).

In contrast we have the Higgs boson discovery with a mass of $m_h = 125.7 \pm 0.3\,(\text{stat.}) \pm 0.3\,(\text{sys.})$ GeV from the LHC [11].

Fortunately there are large loop-corrections or the MSSM would have been excluded already.[18] Because of the size of the Yukawa couplings the largest corrections to the mass come from stop and top loops (see Fig. 4.1 for the relevant Feynman diagrams). In the limit $m_{\tilde{t}_R}, m_{\tilde{t}_L} \gg m_t$, and with stop mass eigenstates close to the chiral eigenstates (more on this later), we get the dominant loop correction:

$$\Delta m_h^2 = \frac{3}{4\pi^2} \cos^2 \alpha \, y_t^2 m_t^2 \ln \left(\frac{m_{\tilde{t}_L} m_{\tilde{t}_R}}{m_t^2} \right), \tag{5.23}$$

where α is a mixing angle for h and H with respect to the superfield component fields H_u^0 and H_d^0, given by

$$\frac{\sin \alpha}{\sin \beta} = -\frac{m_H^2 + m_h^2}{m_H^2 - m_h^2}, \tag{5.24}$$

at tree level.

With this and other corrections the bound is weaker:

$$m_h \leq 135 \text{ GeV},$$

assuming a common sparticle mass scale of $m_{\text{SUSY}} \leq 1\,\text{TeV}$. Higher values for the sparticle masses give large fine-tuning and weaken the bound very little because of the logarithm in Eq. (5.23). The bound can be further weakened by adding extra field content to the MSSM, *e.g.* as in the NMSSM, but for $m_{\text{SUSY}} \approx 1\,\text{TeV}$ there is an upper pertubative limit of $m_h \approx 150\,\text{GeV}$.

It is very interesting to discuss what the Higgs discovery implies for low-energy supersymmetry. As can be seen from the above it requires rather large squark masses even in the favourable scenario with $\tan \beta > 10$. A naive estimate from Eq. (5.23) gives $m_{\tilde{t}} > 1$ TeV. However, this does not take into account negative contributions to the Higgs mass from heavy gauginos, and possible increases in the stop contribution due to tuning of the mixing of the chiral eigenstates in the mass eigenstates.

Since the lightest stop quark is expected to be the lightest squark in scenarios with common GUT scale soft masses—because of the large downward RGE running of m_{33}^Q due to the large top Yukawa coupling—the expected sparticle spectrum lies mostly above 1 TeV, with the possible exception of gauginos/higgsinos. This points to so-called **Split-SUSY** scenarios with heavy scalars and light gauginos, and a relatively large degree of fine-tuning. If one can live with this little hierarchy problem, it will explain why no signs of supersymemtry have been seen yet at the LHC. With squark masses above 1 TeV any hints of SUSY are not likely to come before the machine has been upgraded to 14 TeV in 2014.

If you are willing to accept fine-tuning of the stop mixing instead, or come up with a good reason for why the mixing should be just-so to give a maximal Higgs mass, you can keep fairly light stop quarks. With the addition of light higgsinos and a light gluino the model is then technically natural, these scenarios are called **Natural SUSY** and should be within the current or near future reach of the LHC.

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[18]It is worth pointing out that the MSSM, despite its many parameters, is a falsifiable theory in that had the Higgs boson mass been $\sim 15\,\text{GeV}$ higher, which is allowed in the SM, the MSSM would have been excluded.

In Split-SUSY scenarios with a neutralino dark matter candidate (see below) the lightest neutralino typically has a significant higgsino component. This means that its should be relatively accessible in direct detection experiments due to its large coupling to normal matter, and in the indirect search for neutrinos from captured dark matter annihilation in the Sun. Both types of experiments may very soon see first indications of a signal if this scenario is indeed realised in nature.

To do calculations with the Higgs bosons in the MSSM we need the Feynman rules that result from the relevant Lagrangian terms. Since these have been listed elsewhere we will not repeat them here, but recommend in particular the PhD-thesis of Peter Richardson [12], where they can be found in Appendix A.6, including all interactions with fermions and sfermions. These can also be found, together with all gauge and self-interactions, in the classic paper by Gunion and Haber [13]. Note that in this paper a complex Higgs singlet appears which can safely be ignored.

5.9   The gluino $\tilde{g}$

The **gluino** is a color octet Majorana fermion. As such it has nothing to mix with in the MSSM (even with RPV) and at tree level the mass is given by the soft term M_3. The one complication for the gluino is that it is strongly interacting so $M_3(\mu)$ runs quickly with energy. It is useful to instead talk about the scale-independent **pole-mass**, *i.e.* the pole of the renormalized propagator, $m_{\tilde{g}}$. Including one loop effects due to gluon exchange and squark loops, see Fig. 5.4, in the $\overline{DR}$ scheme we get:

$$m_{\tilde{g}} = M_3(\mu) \left[1 + \frac{\alpha_s}{4\pi} \left(15 + 6 \ln \frac{\mu}{M_3} + \sum_{\text{all } \tilde{q}} A_{\tilde{q}} \right) \right],$$

where the squark loop contributions are

$$A_{\tilde{q}} = \int_0^1 dx\, x \ln \left(x \frac{m_{\tilde{q}}^2}{M_3^2} + (1-x) \frac{m_q^2}{M_e^2} - x(1-x) - i\epsilon \right).$$

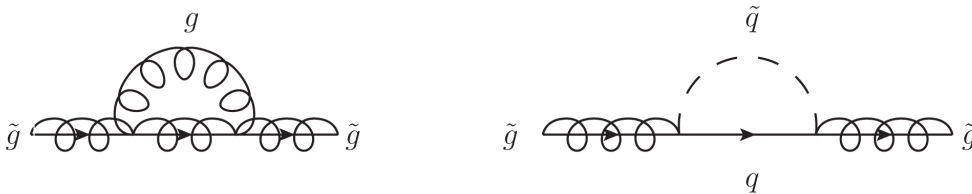Due to the 15-factor the correction can be significant (colour factor).

Figure 5.4: One loop contributions to the gluino mass.

Complete Feynman rules for gluinos can be found in Appendix C of the classic MSSM reference paper of Haber & Kane [14]. A more comprehensible alternative may be Appendix

A.3 from the PhD-thesis of M. Bolz [15]. This also provides a description of how to handle clashing fermion lines that can appear with Majorana fermions.

5.10 Neutralinos & Charginos

We have a bunch of fermion fields that can mix because electroweak symmetry is broken and we do not have to care about $SU(2)_L \times U(1)_Y$ charges, only the $U(1)_{\text{em}}$ charges matter. The candidates are:

$$\tilde{B}^0, \quad \tilde{W}^0, \quad \tilde{W}^\pm, \quad \tilde{H}_u^+, \quad \tilde{H}_u^0, \quad \tilde{H}_d^- \quad \text{and} \quad \tilde{H}_d^0.$$

The only requirement we have is that only fields with equal electromagnetic charge can mix. The neutral (Majorana) gauginos mix as

$$\tilde{\gamma} = N'_{11}\tilde{B}^0 + N'_{12}\tilde{W}^0 \quad \text{(photino)} \tag{5.25}$$
$$\tilde{Z} = N'_{21}\tilde{B}^0 + N'_{22}\tilde{W}^0 \quad \text{(zino)} \tag{5.26}$$

where the mixing is inherited from the gauge boson mixing. More generally, they also mix with the higgsinos to form four **neutralinos**:[19]

$$\tilde{\chi}_i^0 = N_{i1}\tilde{B}^0 + N_{i2}\tilde{W}^0 + N_{i3}\tilde{H}_d^0 + N_{i4}\tilde{H}_u^0. \tag{5.27}$$

In the gauge eigenstate basis

$$\tilde{\chi}^{0T} = \left(\tilde{B}^0, \, \tilde{W}^0, \, \tilde{H}_d^0, \, \tilde{H}_u^0 \right), \tag{5.28}$$

the neutralino mass term can be written as

$$\mathcal{L}_{\chi-\text{mass}} = -\frac{1}{2}\tilde{\chi}^{0T} M_{\tilde{\chi}} \tilde{\chi}^0 + \text{c.c.}$$

where the mass matrix is found from the Lagrangian to be

$$M_{\tilde{\chi}} = \begin{bmatrix} M_1 & 0 & -\frac{1}{\sqrt{2}}g'v_d & \frac{1}{\sqrt{2}}g'v_u \\ 0 & M_2 & \frac{1}{\sqrt{2}}gv_d & -\frac{1}{\sqrt{2}}gv_u \\ -\frac{1}{\sqrt{2}}g'v_d & \frac{1}{\sqrt{2}}gv_d & 0 & -\mu \\ \frac{1}{\sqrt{2}}g'v_u & -\frac{1}{\sqrt{2}}gv_u & -\mu & 0 \end{bmatrix}$$

In this matrix, the upper left diagonal part comes from the soft terms for the $\tilde{B}^0$ and the $\tilde{W}^0$, the lower right off diagonal matrix comes from the superpotential term $\mu H_u H_d$, while the remaining entries come from Higgs-higgsino-gaugino terms from the kinetic part of the Lagrangian, *e.g.* $H_u^\dagger e^{\frac{1}{2}g\sigma W + g'B}H_u$.

---

[19]The neutral higgsinos are also Majorana fermions despite coming from scalar superfields. Unlike the (s)fermion superfields the Higgs superfields have no $\bar{H}$ chiral partners to supply the left-right Weyl spinor combinations required for Dirac fermions. Thus the neutralinos are Majorana fermions.

With the Z-mass condition on the vevs we can also write

$$\frac{1}{\sqrt{2}}g'v_d = \cos\beta\sin\theta_W m_Z, \tag{5.29}$$

$$\frac{1}{\sqrt{2}}g'v_u = \sin\beta\sin\theta_W m_Z, \tag{5.30}$$

$$\frac{1}{\sqrt{2}}gv_d = \cos\beta\cos\theta_W m_Z, \tag{5.31}$$

$$\frac{1}{\sqrt{2}}gv_u = \sin\beta\cos\theta_W m_Z. \tag{5.32}$$

The mass matrix can now be diagonalized to find the $\tilde{\chi}_i^0$ masses.[20] One particularly interesting solution is in the limit where EWSB is a small effect, $m_Z \ll |\mu \pm M_1|$, $|\mu \pm M_2|$, and when $M_1 < M_2 \ll |\mu|$, $\mu \in \mathbb{R}$. Then $\tilde{\chi}_1^0 \approx \tilde{B}^0$, $\tilde{\chi}_2^0 \approx \tilde{W}^0$, $\tilde{\chi}_{3,4}^0 \approx \frac{1}{\sqrt{2}}(\tilde{H}_d^0 \pm \tilde{H}_d^0)$ and

$$m_{\tilde{\chi}_1^0} = M_1 + \frac{m_Z^2 \sin^2\theta_W \sin 2\beta}{\mu} + \dots \tag{5.33}$$

$$m_{\tilde{\chi}_2^0} = M_2 - \frac{m_W^2 \sin 2\beta}{\mu} + \dots \tag{5.34}$$

$$m_{\tilde{\chi}_{3,4}^0} = |\mu| + \frac{m_Z^2}{2\mu}(\operatorname{sgn}\mu \mp \sin 2\beta) + \dots \tag{5.35}$$

Since the LSP is stable in R-parity conserving theories the lightest neutralino is an excellent candidate for dark matter. In particular since a 100 GeV neutralino has a natural relic density close to the measured dark matter density of the Universe. We will return to this issue later.

From the charged fermions we can make **charginos** $\tilde{\chi}_i^\pm$ that are Dirac fermions with mass terms

$$\mathcal{L}_{\chi^\pm - \text{mass}} = -\frac{1}{2}\tilde{\chi}^{\pm T}M_{\chi^\pm}\tilde{\chi}^\pm + \text{c.c.}$$

where $\tilde{\chi}^{\pm T} = (\tilde{W}^+, \tilde{H}_u^+, \tilde{W}^-, \tilde{H}_d^-)$ and

$$M_{\tilde{\chi}^\pm} = \begin{bmatrix} 0 & 0 & M_2 & gv_d \\ 0 & 0 & gv_u & \mu \\ M_2 & gv_u & 0 & 0 \\ gv_d & \mu & 0 & 0 \end{bmatrix}.$$

Here the M_2 terms come from the soft terms for the $W^\pm$, the μ terms come from the superpotential as above, while the remainder come from the kinetic terms. We have

$$gv_d = \sqrt{2}\cos\beta\, m_W, \tag{5.36}$$

$$gv_u = \sqrt{2}\sin\beta\, m_W. \tag{5.37}$$

---

[20]Note that we are perfectly happy with negative or even complex eigenvalues, as this is just a phase for the corresponding mass eigenstate in (5.28). Redefinition of fields can rotate away either the M_1 or M_2 phase, to make the parameter real and positive, but not both and not the μ-phase, which gives rise to problematic CP-violation. Therefore these are often just assumed to be real in order not to violate experimental bounds.

The eigenvalues of this matrix are doubly degenerated (to give the same masses to particles and their anti-particles), and are given as:

$$m_{\tilde{\chi}_{1,2}^{\pm}} = \frac{1}{2}\left(|M_2|^2 + |\mu|^2 + 2m_W^2 \mp \sqrt{(|M_2|^2 + |\mu|^2 + 2m_W^2)^2 - 4|\mu M_2 - m_W^2 \sin^2\beta|^2}\right).$$

In the limit of small EWSB discussed above we have $\tilde{\chi}_1^{\pm} \approx \tilde{W}^{\pm}$ and $\tilde{\chi}_2^{\pm} \approx \tilde{H}_u^+/\tilde{H}_d^-$ with

$$m_{\tilde{\chi}_1^{\pm}} = M_2 - \frac{m_W^2}{\mu}\sin 2\beta, \tag{5.38}$$

$$m_{\tilde{\chi}_2^{\pm}} = |\mu| + \frac{m_W^2}{\mu}\mathrm{sgn}\,\mu. \tag{5.39}$$

Note that in this limit $m_{\tilde{\chi}_2^0} \approx m_{\tilde{\chi}_1^+}$.

We should mention that some authors prefer other symbols for the neutralinos and charginos. Common examples are $\tilde{N}_i$ or $\tilde{Z}_i$ for neutralinos, and $\tilde{C}_i$ or $\tilde{W}_i$ (again!) for charginos.

Feynman rules for charginos & neutralinos can again be found in Haber & Kane [14].

5.11 Sleptons & Squarks

There are multiple contributions to sfermion masses from the MSSM Lagrangian. We make the following list:

i) Under the reasonable assumption that soft masses are (close to) diagonal[21] the sfermions get contributions $-m_F^2\tilde{F}_i^\dagger\tilde{F}_i$ and $-m_f^2\tilde{f}_{iR}^*\tilde{f}_{iR}$ from the soft terms.[22]

ii) There are so-called *hyperfine* terms that come from d-terms $\frac{1}{2}\sum g_a^2(A^*T^aA)^2$ in the scalar potential that give Lagrangian terms of the form (sfermion)^2(Higgs)^2 when one of the scalar fields A is a Higgs field. Under EWSB, when the Higgs field gets a vev these become mass terms. They contribute with a mass

$$\Delta_F = (T_{3F}g^2 - Y_Fg'^2)(v_d^2 - v_u^2) = (T_{3F} - Q_F\sin^2\theta_W)\cos 2\beta\, m_Z^2,$$

where the weak isospin, T_3, hypercharge, Y, and electric charge, Q, are for the left-handed supermultiplet F to which the sfermion belongs. However, these contributions are usually quite small.

iii) There are also so-called F-term contributions that come from Yukawa terms in the superpotential of the form $y_f F H \bar{K}$. From the contribution $\sum|W_i|^2$ to the scalar potential these give Lagrangian terms $y_f^2 H^{0*}H^0\tilde{f}_{iL}^*\tilde{f}_{iL}$ and $y_f^2 H^{0*}H^0\tilde{f}_{iR}^*\tilde{f}_{iR}$. With EWSB we get the mass terms $m_f^2\tilde{f}_{iL}^*\tilde{f}_{iL}$ and $m_f^2\tilde{f}_{iR}^*\tilde{f}_{iR}$ since $m_f = v_{u/d}\,y_f$. These are only significant for large Yukawa coupling y_f.

iv) Furthermore, there are also F-terms that combine scalars from the $\mu H_u H_d$ term and Yukawa terms $y_f F H \bar{K}$ in the superpotential. These give Lagrangian terms $-\mu^* H^{0*}y_f\tilde{f}_L\tilde{f}_R^*$. With a Higgs vev this gives mass terms $-\mu^* v_{u/d}\,y_f\tilde{f}_R^*\tilde{f}_L + \text{c.c.}$

---

[21] This is of course to avoid flavor changing neutral currents (FCNCs).

[22] Here, and in the following, $\tilde{F}_i$ represents an $SU(2)_L$ doublet with generation index i, while $\tilde{f}_{iR}$ represents a singlet.

v) Finally, the soft Yukawa terms of the form $a_f \tilde{F} H \tilde{f}_R^*$ with a Higgs vev give mass terms $a_f v_{u/d} \tilde{f}_L \tilde{f}_R^* + \text{c.c.}$[23]

For the first two generations of sfermions, terms of type iii)–v) are small due to small Yukawa couplings. Then the sfermion masses are *e.g.*

$$m_{\tilde{u}_L}^2 = m_{Q_1}^2 + \Delta\tilde{u}_L, \tag{5.40}$$

$$m_{\tilde{d}_L}^2 = m_{Q_1}^2 + \Delta\tilde{d}_L, \tag{5.41}$$

$$m_{\tilde{u}_R}^2 = m_{u_1}^2 + \Delta\tilde{u}_R. \tag{5.42}$$

Mass splitting between same generation slepton/squark is then given by

$$m_{\tilde{e}_L}^2 - m_{\tilde{\nu}_L}^2 = m_{\tilde{d}_L}^2 - m_{\tilde{u}_L}^2 = -\frac{1}{2}g^2(v_d^2 - v_u^2) = -\cos 2\beta \, m_W^2,$$

since they have the same hypercharge, see Table 5.1. For $\tan\beta > 1$ this gives $m_{\tilde{e}_L}^2 > m_{\tilde{\nu}_L}^2$ and $m_{\tilde{d}_L}^2 > m_{\tilde{u}_L}^2$.

The **third generation sfermions** $\tilde{t}$, $\tilde{b}$ and $\tilde{\tau}$ have a more complicated mass matrix structure, *e.g.* in the gauge eigenstate basis $(\tilde{t}_L, \tilde{t}_R)$ for stop quarks the mass term is

$$\mathcal{L}_{\text{stop}} = -\begin{pmatrix} \tilde{t}_L & \tilde{t}_R \end{pmatrix} m_{\tilde{t}}^2 \begin{pmatrix} \tilde{t}_L \\ \tilde{t}_R \end{pmatrix},$$

where the mass matrix is given by

$$m_{\tilde{t}}^2 = \begin{bmatrix} m_{Q_3}^2 + m_t^2 + \Delta\tilde{u}_L & v(a_t^* \sin\beta - \mu y_t \cos\beta) \\ v(a_t \sin\beta - \mu^* y_t \cos\beta) & m_{u_3}^2 + m_t^2 + \Delta\tilde{u}_R \end{bmatrix}, \tag{5.43}$$

where the diagonal elements come from i), ii) and iii), while the off-diagonal elements come from iv) and v). To find the particle masses, we must diagonalize this matrix, writing it in terms of the mass eigenstates $\tilde{t}_1$ and $\tilde{t}_2$, aquiring also a mixing matrix for the mass eigenstates in terms of the gauge eigenstates $\tilde{t}_L$ and $\tilde{t}_R$:

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = \begin{bmatrix} c_{\tilde{t}} & -s_{\tilde{t}}^* \\ s_{\tilde{t}} & c_{\tilde{t}} \end{bmatrix} \begin{pmatrix} \tilde{t}_L \\ \tilde{t}_R \end{pmatrix}, \tag{5.44}$$

where $m_{\tilde{t}_1}^2 < m_{\tilde{t}_2}^2$ are the eigenvalues of (5.43) and $|c_{\tilde{t}}|^2 + |s_{\tilde{t}}|^2 = 1$. The matrices for $\tilde{b}$ and $\tilde{t}$ have the same structure.

5.12   Gauge coupling unification

We earlier discussed the β-functions for the gauge couplings given in Eq. (4.38). With the MSSM field content we have found and the couplings:[24]

$$g_1 = \sqrt{\frac{5}{3}}g', \qquad g_2 = g, \qquad g_3 = g_s$$

---

[23]We often assume that $a_f = A_0 y_f$ in order to further reduce the FCNC, meaning that there is a global constant A_0 with unit mass relating the Yukawa couplings and the trilinear A-term couplings.

[24]The normalisation choice for g_1 may seem a bit strange, however, this is the correct numerical factor when breaking *e.g.* SU(5) or SO(10) down to the SM group. This factor might be different with a different unified group.

one then gets:

$$b_i^{MSSM} = \left(\frac{33}{5}, 1, -3\right)$$

for

$$\beta_{g_i}\big|_{1-\text{loop}} = \frac{1}{16\pi^2} g_i^3 b_i \tag{5.45}$$

because

$$C(A)_{SU(3)} = 3, \quad C(A)_{SU(2)} = 2, \quad C(A)_{U(1)} = 0$$

from the definition $C(A)\delta_{ij} = (T^a T^b)_{ij}$ and

$$T(R)_{SU(3)} = \frac{1}{2}, \quad T(R)_{SU(2)} = \frac{1}{2}, \quad T(R)_{U(1)} = \frac{3}{5}y^2$$

from the definition $T(R)\delta_{ab} = \text{Tr}\{t_a t_b\}$, e.g. $b_3 = \frac{1}{2} \cdot 12 - 3 \cdot 3 = -3$.

At one loop order we can do a neat rewrite using $\alpha_i \equiv \frac{1}{4\pi} g_i^2$. Since

$$\frac{d}{dt}\alpha_i^{-1} = -2\frac{4\pi}{g_i^3}\frac{d}{dt}g_i$$

we have:

$$\beta_{\alpha_i^{-1}} \equiv \frac{d}{dt}\alpha_i^{-1} = -\frac{8\pi}{g_i^3}\frac{1}{16\pi^2}g_i^3 b_i = -\frac{b_i}{2\pi}$$

Thus α^{-1} runs linearly in t at one loop. By running from the EW scale measured values to high scale it is observed that in the MSSM the coupling constants unify, which they do not naturally do in the SM. The unification scale (GUT) is $m_{GUT} \approx 2 \cdot 10^{16}$ GeV, about two orders of magnitude below the Planck scale. See Fig. 5.5, taken from Martin [16].

Something funny happens to the gaugino mass parameters. Their β functions are

$$\beta_{M_i}\big|_{1-\text{loop}} \equiv \frac{d}{dt}M_i = \frac{1}{8\pi^2}g_i^2 M_i b_i$$

Thus all three ratios M_i/g_i^2 are scale independent at one loop. To see this let $R = M_i/g_i^2$ then

$$\beta_R = \frac{\frac{d}{dt}M_i g_i^2 - M_i \frac{d}{dt}g_i^2}{g_i^4} = \frac{\frac{1}{8\pi^2}g_i^2 M_i b_i \cdot g_i^2 - M_i \cdot 2g_i \cdot \frac{1}{16\pi}g_i^3 b_i}{g_i^4} = 0 \tag{5.46}$$

If we now assume the coupling constants unify at some high scale m_{GUT} to the coupling g_u, and that the gauginos have a common mass $m_{1/2} = M_1(m_{GUT}) = M_2(m_{GUT}) = M_3(m_{GUT})$, it follows that

$$\frac{M_1}{g_1^2} = \frac{M_2}{g_2^2} = \frac{M_3}{g_3^2} = \frac{m_{1/2}}{g_u^2}$$

at all scales! This is a very powerful and predictive assumption. It leads to

$$M_3 = \frac{\alpha_s}{\alpha}\sin^2\theta_W M_2 = \frac{3}{5}\frac{\alpha_s}{\alpha}\cos^2\theta_W M_1 \tag{5.47}$$

which numerically predicts

$$M_3 : M_2 : M_1 = 6 : 2 : 1$$

at $\mu = 1$ TeV. Comparing to our discussion for neutralinos and charginos this predicts $m_{\tilde{g}} = 6m_{\tilde{\chi}_1^0}$, $m_{\tilde{\chi}_2^0} \approx m_{\tilde{\chi}_1^\pm} = 2m_{\tilde{\chi}_1^0}$. However, it is important to remember that this often used relationship is based on the *conjecture* of gauge coupling unification!
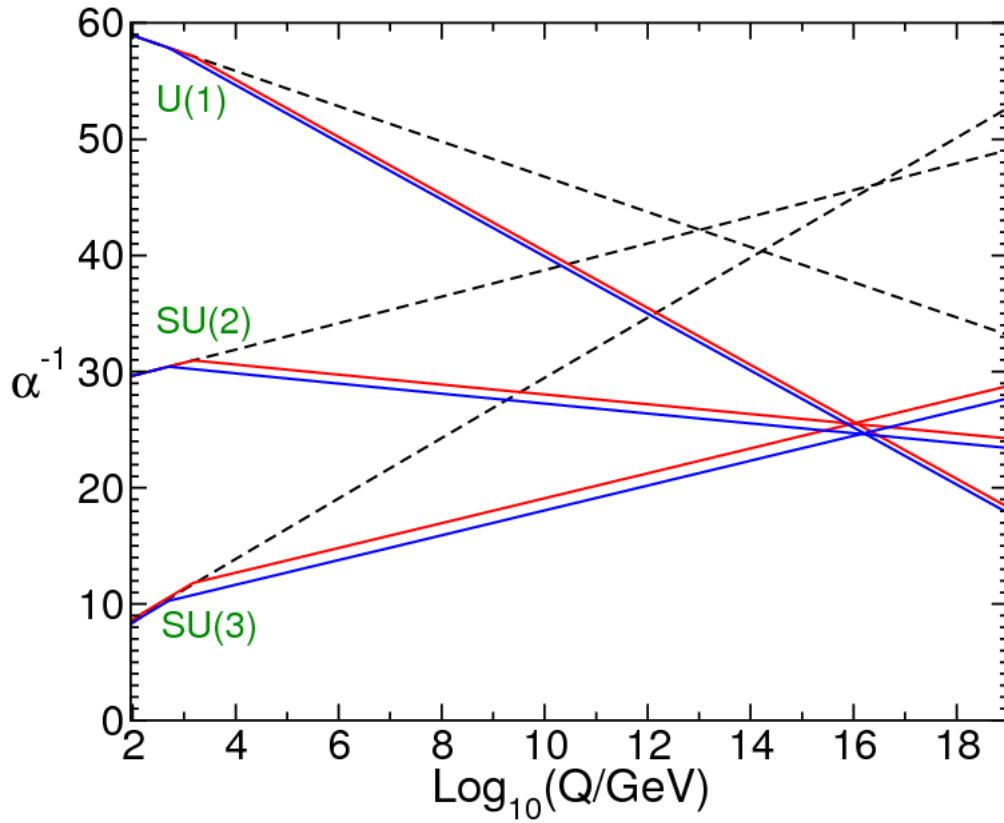
Figure 5.5: RG evolution of the inverse gauge couplings $\alpha_a^{-1}(Q)$ in the SM (dashed lines) and the MSSM (solid lines). In the MSSM case, the sparticle mass thresholds are varied between 250 GeV and 1 TeV and $\alpha_3(m_Z)$ between 0.113 and 0.123. Two-loop effects are included.

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