Lecture notes for FYS5190/FYS9190 – Supersymmetry

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Chapter 1

Groups and algebras

The goal of these lectures is to introduce the basics of low-energy models of supersymmetry (SUSY) using the Minimal Supersymmetric Standard Model (MSSM) as a main example. Rather than starting with the problems of the SM, we will focus on the algebraic origin of SUSY in the sense of an extension of the symmetries of Einsten's Special Relativity (SR), which was the original motivation for SUSY.

1.1 What is a group?

> **Definition**: The set $G = \{g_i\}$ and operation $\bullet$ form a **group** if and only if for $\forall\, g_i \in G$
>
> i) $g_i \bullet g_j \in G$ (closure)
>
> ii) $(g_i \bullet g_j) \bullet g_k = g_i \bullet (g_j \bullet g_k)$ (associativity)
>
> iii) $\exists e \in G$ such that $g_i \bullet e = e \bullet g_i = g_i$ (identity element)
>
> iv) $\exists g_i^{-1} \in G$ such that $g_i \bullet g_i^{-1} = g_i^{-1} \bullet g_i = e$ (inverse)

A simple example of a group is $G = \mathbb{Z}$ with usual addition as the operation, $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to $\mathbb{Z}_n$, where the operation is addition with modulo n. In this group, $g_i^{-1} = n - g_i$ and the unit element is $e = 0$. Note that $\mathbb{Z}$ is an *infinite* group, while $\mathbb{Z}_n$ is finite, with *order n* (meaning n members). Both are *abelian* groups, meaning that $g_i \bullet g_j = g_j \bullet g_i$.

All of this is "only" mathematics. Physicists are often more interested in groups where the elements of G *act* on some elements of a set $s \in S$, $g(s) = s' \in S$.[1] S here can for example be the state of a system, say a wave-function in quantum mechanics. We will return to this in a moment, let us just mention that the operation $g_i \bullet g_j$ acts as $(g_i \bullet g_j)(s) = g_i \bullet (g_j(s))$ and the identity acts as $e(s) = s$.[2]

[1] As a result mathematics courses in group theory are not always so relevant to a physicist.

[2] We can prove this from iii) in the definition. Note that we use e as the identity in an abstract group, while

A more sophisticated example of a group can be found in a use for the Taylor expansion[3]

$$\begin{aligned} f(x+a) &= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \ldots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{dx^n} f(x) \\ &= e^{a\frac{d}{dx}} f(x) \end{aligned}$$

The operator $T_a = e^{a\frac{d}{dx}}$ is called the **translation operator** (in this case in one dimension). Together with the operation $T_a \bullet T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where $T_a^{-1} = T_{-a}$. In N dimensions the group $T(N)$ has the elements $T_{\vec{a}} = e^{\vec{a}\cdot\vec{\nabla}}$.

Definition: A subset $H \subset G$, is a **subgroup** if and only if:[a]

i) $h_i \bullet h_j \in H$ for $\forall h_i, h_j \in H$

ii) $h_i^{-1} \in H$ for $\forall h_i \in H$

[a]An alternative, more compact, way of writing these two requirements is $h_i \bullet h_j^{-1} \in H$ for $\forall h_i, h_j \in G$. This is often utilised in proofs.

Definition: H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$. A subgroup H is a **normal** (invariant) subgroup, if and only if for $\forall g \in G$,

$$ghg^{-1} \in H \text{ for } \forall h \in H$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no abelian normal subgroup.

The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U, i.e. matrices such that $U^\dagger U = 1$ or $U^{-1} = U^\dagger$. This has the neat property that for $\forall \vec{x}, \vec{y} \in \mathbb{C}^n$ multiplication by a unitary matrix leaves scalar products unchanged:

$$\begin{aligned} \vec{x}' \cdot \vec{y}' &\equiv \vec{x}'^\dagger \vec{y}' = (U\vec{x})^\dagger U\vec{y} \\ &= \vec{x}^\dagger U^\dagger U \vec{y} = \vec{x}^\dagger \vec{y} = \vec{x} \cdot \vec{y} \end{aligned}$$

If we additionally require that $\det(U) = 1$ the matrices form the **special unitary group** $SU(n)$. Let $U_i, U_j \in SU(n)$, then

$$\det(U_i U_j^{-1}) = \det(U_i)\det(U_j^{-1}) = 1.$$

1 is used as the identity matrix in matrix representations.

[3]This is the first of many points where any real mathematician would start to cry loudly and leave the room.

This means that $U_i U_j^{-1} \in SU(N)$. In other words, $SU(n)$ is a **proper subgroup** of $U(n)$. Let $V \in U(n)$ and $U \in SU(n)$, then $VUV^{-1} \in SU(n)$ because:

$$\det(VUV^{-1}) = \det(V)\det(U)\det(V^{-1}) = \frac{\det(V)}{\det(V)}\det(U) = 1.$$

In other words, $SU(n)$ is also a **normal subgroup** of $U(n)$.

Definition: A **(left) coset** of a subgroup $H \subset G$ is a set $\{gh : h \in H\}$ where $g \in G$ and a **(right) coset** of a subgroup $H \subset G$ is a set $\{hg : h \in H\}$ where $g \in G$. For normal subgroups H the left and right cosets coincide and form the **coset group** G/H which has the members $\{gh : h \in H\}$ for $\forall g \in G$ and the binary operation $*$ with $gh * g'h' \in \{(g \bullet g')h : h \in H\}$.

Definition: The **direct product** of groups G and H, $G \times H$, is defined as the *ordered pairs* (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet h_j)$. $G \times H$ is then a group and G and H are normal subgroups of $G \times H$.

Definition: The **semi-direct product** $G \rtimes H$, where G is a mapping $G : H \to H$, is defined by the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet g_i(h_j))$. Here H is not a normal subgroup of $G \rtimes H$.

The SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is an example of a direct product. Direct products are "trivial" structures because there is no "interaction" between the subgroups. Can we imagine a group $G \supset SU(3)_c \times SU(2)_L \times U(1)_Y$ that can be broken down to the SM group but has a non-trivial unified gauge structure? There is, $SU(5)$ being one example.

1.2 Representations

Definition: A **representation** of a group G on a vector space V is a map $\rho : G \to GL(V)$, where $GL(V)$ is the **general linear group** on V, i.e. invertible matrices of the field of V, such that for $\forall g_i, g_i \in G$, $\rho(g_i g_j) = \rho(g_i)\rho(g_j)$ (homeomorphism).

For $U(1)$ the transformation $e^{i\chi\alpha}$ is the **fundamental** or **defining representation** which can be used on wavefunctions $\psi(x)$—these form a one dimensional vector space over the complex numbers. For $SU(2)$ the transformation $e^{i\alpha_i\sigma_i}$, with σ being the Pauli matrices, is the **fundamental representation**, which can be applied to *e.g.* weak doublets $\psi = (\nu_l, l)$.[4]

[4]This is a bit daft, since both $U(1)$ and $SU(2)$ are defined in terms of matrices. However, we will also have use for other representations, e.g. the **adjoint representation**, which is not the fundamental or defining representation.

Definition: Two representations ρ and ρ' of G on V and V' are **equivalent** if and only if $\exists A : V \to V'$, that is one-to-one, such that for $\forall g \in G$, $A\rho(g)A^{-1} = \rho'(g)$.

Definition: An **irreducible representation** ρ is a representation where there is *no* proper subspace $W \subset V$ that is closed under the group, i.e. there is no $W \subset V$ such that for $\forall w \in W$, $\forall g \in G$ we have $\rho(g)w \in W$.[a]

[a] In other words, we can not split the matrix representation of G in two parts that do not "mix".

Let $\rho(g)$ for $g \in G$ act on a vector space V as a matrix. If $\rho(g)$ can be decomposed into $\rho_1(g)$ and $\rho_2(g)$ such that

$$\rho(g)v = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix} v$$

for $\forall v \in V$, then ρ is **reducible**.

Definition: $T(R)$ is the **Dynkin index** of the representation R in terms of matrices T_a, given by $\mathrm{Tr}[T_a, T_b] = T(R)\delta_{ab}$. $C(R)$ is the **Casimir invariant** given by $C(R)\delta_{ij} = (T^aT^a)_{ij}$

1.3 Lie groups

We begin by defining what we mean by Lie groups

Definition: A **Lie group** G is a finite-dimensional, n, **smooth manifold** C^∞, i.e. for $\forall g \in G$, g can locally be mapped onto (parametrised by) $\mathbb{R}^n$ or $\mathbb{C}^n$, and group multiplication and inversion are smooth functions, meaning that given $g(\vec{a}), g'(\vec{a}) \in G$, $g(\vec{a}') \bullet g'(\vec{a}') = g''(\vec{b})$ where $\vec{b}(\vec{a}, \vec{a}')$ is analytic, and $g^{-1}(\vec{a}) = g'(\vec{a}')$ where $\vec{a}'(\vec{a})$ is analytic.

In terms of a Lie group G acting on a vector space V, $\dim(V) = m$ (or more generally an m-dimensional manifold), this means we can write the map $G \times V \to V$ for $\vec{x} \in V$ as $x_i \to x_i' = f_i(x_i, a_j)$ where f_i is analytic in x_i and a_j. Additionally f_i should have an inverse.

The translation group $T(1)$ with $g(a) = e^{a\frac{d}{dx}}$ is a Lie group since $g(a) \cdot g(a') = g(a + a')$ and $a + a'$ is analytic. Here we can write $f(x, a) = x + a$. $SU(n)$ are Lie groups as they have a fundamental representation $e^{i\vec{\alpha}\vec{\lambda}}$ where λ is a set of $n \times n$-matrices, and $f_i(\vec{x}, \vec{\alpha}) = [e^{i\vec{\alpha}\vec{\lambda}}\vec{x}]_i$.

By the analyticity we can always construct the parametrization so that $g(0) = e$ or $x_i =$

$f_i(x_i, 0)$. By an infinitesimal transformation da_i we then get the following Taylor expansion[5]

$$\begin{aligned} x_i' &= x_i + dx_i = f_i(x_i, da_i) \\ &= f_i(x_i, 0) + \frac{\partial f_i}{\partial a_j} da_j + \ldots \\ &= x_i + \frac{\partial f_i}{\partial a_j} da_j \end{aligned}$$

This is the transformation by the member of the group that in the parameterisation sits da_j from the identity. If we now let F be a function from the vector space V to either the real $\mathbb{R}$ or complex numbers $\mathbb{C}$, then the group transformation defined by da_i changes F by

$$\begin{aligned} dF &= \frac{\partial F}{\partial x_i} dx_i \\ &= \frac{\partial F}{\partial x_i} \frac{\partial f_i}{\partial a_j} da_j \\ &\equiv da_j X_j F \end{aligned}$$

where the operators defined by

$$X_j \equiv \frac{\partial f_i}{\partial a_j} \frac{\partial}{\partial x_i}$$

are called the n **generators** of the Lie group. It is these generators X that define the action of the Lie group in a given representation as the a's are mere parameters.

As an example of the above we can now go in the opposite direction and look at the two-parameter transformation *defined* by

$$x' = f(x) = a_1 x + a_2,$$

which gives

$$X_1 = \frac{\partial f}{\partial a_1} \frac{\partial}{\partial x} = x \frac{\partial}{\partial x},$$

which is the generator for **dilation** (scale change), and

$$X_2 = \frac{\partial}{\partial x},$$

which is the generator for $T(1)$. Note that $[X_1, X_2] = -X_2$.

Exercise: Find the generators of $SU(2)$ and their commutation relationships. Hint: One answer uses the Pauli matrices, but try to derive this from an infinitesimal parametrization.

Next we lists three central results on Lie groups derived by Sophus Lie [6]:

[5] The fact that f_i is analytic means that this Taylor expansion must converge in some radius around $f_i(x_i, 0)$.

Theorem: (Lie's theorems)

i) For a Lie group $\frac{\partial f_i}{\partial a_j}$ is analytic.

ii) The generators X_i satisfy $[X_i, X_j] = C_{ij}^k X_k$, where C_{ij}^k are **structure constants**.

iii) $C_{ij}^k = -C_{ji}^k$ and $C_{ij}^k C_{kl}^m + C_{jl}^k C_{ki}^m + C_{li}^k C_{kj}^m = 0$.[a]

[a]The second identity follows from the Jacobi identity $[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0$

Exercise: What are the structure constants of SU(2)?

1.4 Lie algebras

Definition: An **algebra** A on a field (say $\mathbb{R}$ or $\mathbb{C}$) is a linear vector space with a binary operation $\circ : A \times A \to A$.

The vector space $\mathbb{R}^3$ together with the cross-product constitutes an algebra.

Definition: A **Lie algebra** L is an algebra where the binary operator $[\,,\,]$, called Lie bracket, has the properties that for $x, y, z \in L$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$):

i) (associativity)

$$[ax + by, z] = a[x, z] + b[y, z]$$

$$[z, ax + by] = a[z, x] + b[z, y]$$

ii) (anti-commutation)

$$[x, y] = -[y, x]$$

iii) (Jacobi identity)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

We usually restrict ourselves to algebras of linear operators with $[x, y] = xy - yx$, where property iii) is automatic. From Lie's theorems the generators of an n-dimensional Lie group form an n-dimensional Lie algebra.

We mentioned the fundamental representation of a matrix based group earlier. These representations have the lowest possible dimension. Another important representation is the **adjoint**. This consists of the matrices:

$$(M_i)_j^k = -C_{ij}^k$$

where C_{ij}^k are the structure constants. From the Jacobi identity we have $[M_i, M_j] = C_{ij}^k M_k$, meaning that the adjoint representation fulfills the same algebra as the fundamental (generators). Note that the dimension of the fundamental representation n for $SO(n)$ and $SU(n)$ is always smaller than the adjoint, which is equal to the degrees of freedom, $\frac{1}{2}n(n-1)$ and n^2-1 respectively.

Exercise: Find the dimensions of the fundamental and adjoint representations of $SU(n)$.

Exercise: Find the fundamental representation for $SO(3)$ and the adjoint representation for $SU(2)$. What does this say about the groups and their algebras?

Chapter 2

The Poincaré algebra and its extensions

We now take a look at the groups behind Special Relativity (SR), the Lorentz and Poincaré groups, and look for ways to extend them to internal symmetries, *i.e.* gauge groups.

2.1 The Lorentz Group

A point in the Minkowski space-time manifold $\mathbb{M}_4$ is given by $x^\mu = (t, x, y, z)$ and Einstein's requirement was that physics should be invariant under the Lorentz group.

> **Definition**: The **Lorentz group** L is the group of linear transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that $x^2 = x_\mu x^\mu = x'_\mu x'^\mu$ is invariant. The **proper orthochronous Lorentz group** $L_+^\uparrow$ is a subgroup of L where $\det\Lambda = 1$ and $\Lambda^0{}_0 \geq 1$. [a]
>
> [a] This guarantees that time moves forward, and makes space and time reflections impossible, with the group describing only boosts and rotations.

From the discussion in the previous section one can show that any $\Lambda \in L_+^\uparrow$ can be written as

$$\Lambda^\mu{}_\nu = \left[\exp\left(-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)\right]^\mu{}_\nu, \tag{2.1}$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ are the parameters of the transformation and $M_{\rho\sigma}$ are the generators of L, and the basis of the Lie algebra for L, and are given by:

$$M = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix},$$

where K_i and J_i are generators of boost and rotation respectively. These fulfil the following algebra:[1]

$$[J_i, J_j] = -i\epsilon_{ijk}J_k, \tag{2.2}$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k, \tag{2.3}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.4}$$

The generators M of L obey the commutation relation:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \tag{2.5}$$

2.2 The Poincaré group

We extend L by translation to get the Poincaré group, where translation : $x^\mu \to x'^\mu = x^\mu + a^\mu$. This leaves lengths $(x-y)^2$ invariant in $\mathbb{M}_4$.

> **Definition**: The **Poincaré group** P is the group of all transformations of the form
>
> $$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$
>
> We can also construct the **restricted Poincaré group** $P_+^\uparrow$, by restricting the matrices Λ in the same way as in $L_+^\uparrow$.

We see that the composition of two elements in the group is:

$$(\Lambda_1, a_1) \bullet (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1).$$

This tells us that the Poincaré group is **not** a direct product of the Lorentz group and the translation group, but a **semi-direct product** of L and the translation group $T(1,3)$, $P = L \rtimes T(1,3)$. The translation generators P_μ have a trivial commutation relationship:[2]

$$[P_\mu, P_\nu] = 0 \tag{2.6}$$

One can show that:[3]

$$[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \tag{2.7}$$

Equations (2.5)–(2.7) form the **Poincaré algebra**, a Lie algebra.

2.3 The Casimir operators of the Poincaré group

> **Definition**: The **Casimir operators** of a Lie algebra are the operators that commute with all elements of the algebra [a]
>
> [a] Technically we say they are members of the centre of the universal enveloping algebra of the Lie algebra. Whatever that means.

[1] Notice that (2.2) and (2.4) are the $SU(2)$ algebra.

[2] This means that the translation group in Minkowski space is abelian. This is obvious, since $x^\mu + y^\mu = y^\mu + x^\mu$. One can show that the differential representation is the expected $P_\mu = -i\partial_\mu$.

[3] For a rigorous derivation of this see Chapter 1.2 of [8]

A central theorem in representation theory for groups and algebras is **Schur's lemma**:

Theorem: (Schur's Lemma)
In any irreducible representation of a Lie algebra, the Casimir operators are proportional to the identity.

This has the wonderful consequence that the constants of proportionality can be used to classify the (irreducible) representations of the Lie algebra (and group). Let us take a concrete example to illustrate: $P^2 = P_\mu P^\mu$ is a Casimir operator of the Poincaré algebra because the following holds:[4]

$$\left[P_\mu, P^2\right] = 0, \tag{2.11}$$
$$\left[M_{\mu\nu}, P^2\right] = 0. \tag{2.12}$$

This allows us to label the irreducible representation of the Poincaré group with a quantum number m^2, writing a corresponding state as $|m\rangle$, such that:[5]

$$P^2|m\rangle = m^2|m\rangle.$$

The number of Casimir operators is the **rank** of the algebra, *e.g.* rank $SU(n) = n-1$. It turns out that $P_+^\uparrow$ has rank 2, and thus two Casimir operators. To demonstrate this is rather involved, and we won't make an attempt here, but note that it can be shown that[6] $L_+^\uparrow \cong SU(2) \times SU(2)$ because of the structure of the boost and rotation generators, where $SU(2)$ can be shown to have rank 1. Furthermore, $L_+^\uparrow \cong SL(2,\mathbb{C})$. We will return to this relationship between $L_+^\uparrow$ and $SL(2,\mathbb{C})$ in Section 2.5, where we use it to reformulate the algebras we work with in supersymmetry.

So, what is the second Casimir of the Poincaré algebra?

Definition: The **Pauli-Ljubanski polarisation vector** is given by:

$$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{2.13}$$

[4] The first relation follows trivially from the commutation of P_μ with P_ν. To show the second we first use that

$$[M_{\mu\nu}, P_\rho P^\rho] = [M_{\mu\nu}, P_\rho]P^\rho + P_\rho[M_{\mu\nu}, P^\rho], \tag{2.8}$$

and Eq. (2.7) to get:

$$[M_{\mu\nu}, P_\rho P^\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu)P^\rho - iP_\rho(g_\mu{}^\rho P_\nu - g_\nu{}^\rho P_\mu), \tag{2.9}$$

thus

$$[M_{\mu\nu}, P_\rho P^\rho] = -2i[P_\mu, P_\nu] = 0. \tag{2.10}$$

[5] This quantum number looks astonishingly like mass and P^2 like the square of the 4-momentum operator. However, we note that in general m^2 is not restricted to be larger than zero.

[6] Here $\cong$ means homomorfic, that is structure preserving.

Then $W^2 = W_\mu W^\mu$ is a Casimir operator of $P_+^\uparrow$, *i.e.*:

$$\left[M_{\mu\nu}, W^2\right] = 0 \tag{2.14}$$
$$\left[P_\mu, W^2\right] = 0 \tag{2.15}$$

To show this we can re-write the operator as:[7]

$$W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M^{\rho\sigma}M_{\nu\sigma}P_\rho P^\nu.$$

From the above it is easy to show that W^2 is indeed a Casimir

Again, because W^2 is a Casimir operator, we can label all states in an irreducible representation (read particles) with quantum numbers m, s, such that:

$$W^2|m,s\rangle = -m^2 s(s+1)|m,s\rangle$$

The m^2 appears because there are two P_μ operators in each term. However, what is the significance of the s, and why do we choose to write the quantum number in that (familiar?) way? One can easily show using ladder operators that $s = 0, \frac{1}{2}, 1, \ldots$, *i.e.* can only take integer and half integer values. In the rest frame (RF) of the particle we have:[8]

$$P_\mu = (m, \vec{0})$$

Using that $WP = 0$ this gives us $W_0 = 0$ in the RF, and furthermore:

$$W_i = \frac{1}{2}\epsilon_{i0jk} m M^{jk} = m S_i,$$

where $S_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ is the **spin operator**. This gives $W^2 = -\vec{W}^2 = -m^2\vec{S}^2$, meaning that s is indeed the spin quantum number.[9]

The conclusion of this subsection is that anything transforming under the Poincaré group, meaning the objects considered by SR, can be classified by two quantum numbers: mass and spin.

2.4 The no-go theorem and graded Lie algebras

Since we now know the Poincaré group and its representations well, we can ask: Can the external space-time symmetries be extended, perhaps also to include the internal gauge symmetries? Unfortunately no. In 1967 Coleman and Mandula [2] showed that any extension of the Pointcaré group to include gauge symmetries is isomorphic to $G_{SM} \times P_+^\uparrow$, *i.e.* the generators B_i of standard model gauge groups all have

$$[P_\mu, B_i] = [M_{\mu\nu}, B_i] = 0.$$

Not to be defeated by a simple mathematical proof this was countered by Haag, Łopuszański and Sohnius (HLS) in 1975 in [5] where they introduced the concept of graded Lie algebras

[7] This is non-trivial to demonstrate, see Chapter 1.2 of [8].

[8] This does not loose generality since physics should be independent of frame.

[9] Observe that this discussion is problematic for massless particles. However, it is possible to find a similar relation for massless particles, when we chose a frame where the velocity of the particle is mono-directional.

to get around the no-go theorem.

> **Definition**: A ($\mathbb{Z}_2$) **graded Lie algebra** or **superalgebra** is a vector space L that is a direct sum of two vector spaces L_0 and L_1, $L = L_0 \oplus L_1$ with a binary operation $\bullet : L \times L \to L$ such that for $\forall x_i \in L_i$
>
> i) $x_i \bullet x_j \in L_{i+j \mod 2}$ (grading)[a]
>
> ii) $x_i \bullet x_j = -(-1)^{ij} x_j \bullet x_i$ (supersymmetrization)
>
> iii) $x_i \bullet (x_j \bullet x_k)(-1)^{ik} + x_j \bullet (x_k \bullet x_i)(-1)^{ji} + x_k \bullet (x_i \bullet x_j)(-1)^{kj} = 0$ (generalised Jacobi identity)
>
> This definition can be generalised to $\mathbb{Z}_n$ by a direct sum over n vector spaces L_i, $L = \oplus_{i=0}^{n-1} L_i$, such that $x_i \bullet x_j \in L_{i+j \mod n}$ with the same requirements for supersymmetrization and Jacobi identity as for the $\mathbb{Z}_2$ graded algebra.
>
> [a]This means that $x_0 \bullet x_0 \in L_0$, $x_1 \bullet x_1 \in L_0$ and $x_0 \bullet x_1 \in L_1$.

We can start, as HLS, with a Lie algebra ($L_0 = P_+^\uparrow$) and add a new vector space L_1 spanned by four operators, the Majorana spinor charges Q_a. It can be shown that the superalgebra requirements are fulfilled by:

$$[Q_a, P_\mu] = 0 \tag{2.16}$$
$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu} Q)_a \tag{2.17}$$
$$\{Q_a, \overline{Q}_b\} = 2\not{P}_{ab} \tag{2.18}$$

where $\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and as usual $\overline{Q}_a = (Q^\dagger \gamma_0)_a$.[10]

Unfortunately, the internal gauge groups are nowhere to be seen. They can appear if we extend the algebra with Q_a^α, where $\alpha = 1, \ldots, N$, which gives gives rise to so-called $N > 1$ supersymmetries. This introduces extra particles and does not seem to be realised in nature due to an extensive number of extra particles.[11] This extension, including $N > 1$, can be proven, under some reasonable assumptions, to be the **largest possible** extension of SR.

2.5 Weyl spinors

Previously we claimed that there is a homomorphism between $L_+^\uparrow$ and $SL(2, \mathbb{C})$. This homomorphism, with $\Lambda^\mu{}_\nu \in L_+^\uparrow$ and $M \in SL(2, \mathbb{C})$, can be explicitly given by:[12]

$$\Lambda^\mu{}_\nu(M) = \frac{1}{2}\mathrm{Tr}[\overline{\sigma}^\mu M \sigma_\nu M^\dagger], \tag{2.19}$$
$$M(\Lambda^\mu{}_\nu) = \pm\frac{1}{\sqrt{\det(\Lambda^\mu{}_\nu \sigma_\mu \overline{\sigma}^\nu)}}\Lambda^\mu{}_\nu \sigma_\mu \overline{\sigma}^\nu, \tag{2.20}$$

where $\overline{\sigma}^\mu = (1, -\vec{\sigma})$ and $\sigma^\mu = (1, \vec{\sigma})$.

[10]Alternatively, (2.18) can be written as $\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu$.

[11]Note that $N > 8$ would include particles with spin greater than 2.

[12]The sign in Eq. (2.20) is the reason that this is a homomorphism, instead of an isomorphism. Each element in $SL(2, \mathbb{C})$ can be assigned to two in $L_+^\uparrow$.

Since we have this homomorphism we can look at the representations of $SL(2,\mathbb{C})$ instead of the Poincaré group (with its usual Dirac spinors) when we describe particles, but what are those representations? It turns out that there exist two inequivalent fundamental representations of $SL(2,\mathbb{C})$:

i) The self-representation $\rho(M) = M$ working on an element ψ of a representation space F:

$$\psi'_A = M_A{}^B \psi_B \qquad A, B = 1, 2$$

ii) The complex conjugate self-representation $\rho(M) = M^*$ working on $\overline{\psi}$ in a space $\dot{F}$:[13]

$$\overline{\psi}'_{\dot{A}} = (M^*)_{\dot{A}}{}^{\dot{B}} \overline{\psi}_{\dot{B}} \qquad \dot{A}, \dot{B} = 1, 2$$

Definition: ψ and $\overline{\psi}$ are called **left- and right-handed Weyl spinors**.

Indices can be lowered and raised with:

$$\epsilon_{AB} = \epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\epsilon^{AB} = \epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The relationship between ψ and $\overline{\psi}$ can be expressed with:[14]

$$\overline{\sigma^0}^{\dot{A}A} (\psi_A)^* = \overline{\psi}^{\dot{A}}$$

Note that from the above:

$$(\psi_A)^\dagger = \overline{\psi}_{\dot{A}}$$
$$(\overline{\psi}_{\dot{A}})^\dagger = \psi_A$$

We define contractions of Weyl spinors as follows:

Definition: $\psi\chi \equiv \psi^A \chi_A$ and $\overline{\psi}\overline{\chi} \equiv \overline{\psi}_{\dot{A}} \overline{\chi}^{\dot{A}}$.

These quantities are invariant under $SL(2,\mathbb{C})$. With this in hand we see that

$$\psi^2 \equiv \psi\psi = \psi^A \psi_A = \epsilon^{AB} \psi_B \psi_A = \epsilon^{12} \psi_2 \psi_1 + \epsilon^{21} \psi_1 \psi_2 = \psi_2 \psi_1 - \psi_1 \psi_2.$$

This quantity is zero if the Weyl spinors commute. In order to avoid this we make the following assumption which is consistent with how we treat fermions (and Dirac spinors):

Postulate: All Weyl spinors anticommute:[a] $\{\psi_A, \psi_B\} = \{\overline{\psi}_{\dot{A}}, \overline{\psi}_{\dot{B}}\} = \{\psi_A, \overline{\psi}_{\dot{B}}\} = \{\overline{\psi}_{\dot{A}}, \psi_B\} = 0$.

[a] This means that Weyl spinors are so-called **Grassmann numbers**.

[13] The dot on the indices is just there to help us remember which sum is which and does not carry any additional importance.

[14] This is a bit daft, as $\overline{\sigma^0}^{\dot{A}A} = \delta_{\dot{A}A}$, and we will in the following omit the matrix and write $(\psi_A)^* = \overline{\psi}^{\dot{A}}$.

This means that

$$\psi^2 \equiv \psi\psi = \psi^A\psi_A = -2\psi_1\psi_2.$$

Weyl spinors can be related to Dirac spinors ψ_a as well:[15]

$$\psi_a = \begin{pmatrix} \psi_A \\ \overline{\chi}^{\dot{A}} \end{pmatrix}.$$

We see that in order to describe a Dirac spinor we need both handedness of Weyl spinor. For Majorana spinors we have:

$$\psi_a = \begin{pmatrix} \psi_A \\ \overline{\psi}^{\dot{A}} \end{pmatrix}.$$

We can now write the super-Poincaré algebra (superalgebra) in terms of Weyl spinors. With

$$Q_a = \begin{pmatrix} Q_A \\ \bar{Q}^{\dot{A}} \end{pmatrix}, \tag{2.21}$$

for the Majorana spinor charges, we have

$$\{Q_A, Q_B\} = \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\} = 0 \tag{2.22}$$
$$\{Q_A, \overline{Q}_{\dot{B}}\} = 2\sigma^{\mu}_{A\dot{B}}P_\mu \tag{2.23}$$
$$[Q_A, P_\mu] = [\overline{Q}_{\dot{A}}, P_\mu] = 0 \tag{2.24}$$
$$[Q_A, M^{\mu\nu}] = \sigma^{\mu\nu}{}_A{}^B Q_B \tag{2.25}$$

where $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)$.

Exercise: Show that $L_+^\uparrow$ and $SL(2,\mathbb{C})$ are indeed **homomorphic**, *i.e.* that the mapping defined by (2.19) or (2.20) has the property that $\Lambda(M_1M_2) = \Lambda(M_1)\Lambda(M_2)$ or $M(\Lambda_1\Lambda_2) = M(\Lambda_1)M(\Lambda_2)$.

2.6 The Casimir operators of the super-Poincaré algebra

When Q_a is four-dimensional it is easy to see that P^2 is still a Casimir operator of the superalgebra. From Eq. (2.24) P_μ commutes with the Qs, so in turn P^2 must commute. However, W^2 is not a Casimir because of the following result:

$$[W^2, Q_a] = W_\mu(\not{P}\gamma_\mu\gamma^5 Q)_a + \frac{3}{4}P^2Q_a.$$

We want to find an extension of W that commutes with the Qs while retaining the commutators we alread have. The construction

$$C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu,$$

[15] Note that in general $(\psi_A)^* \neq \overline{\chi}^{\dot{A}}$.

where

$$B_\mu \equiv W_\mu + \frac{1}{4}X_\mu,$$

with

$$X_\mu \equiv \frac{1}{2}\overline{Q}\gamma_\mu\gamma^5 Q,$$

has the required relation:

$$[C_{\mu\nu}, Q_a] = 0.$$

By excessive algebra we can show that:

$$\begin{aligned}
[C^2, Q_a] &= 0 \quad \text{(trivial)} \\
[C^2, P_\mu] &= 0 \quad \text{(algebra)} \\
[C^2, M_{\mu\nu}] &= 0 \quad \text{(because } C^2 \text{ is a Lorentz scalar)}
\end{aligned}$$

Thus C^2 is a Casimir operator for the superalgebra.

2.7 Representations of the superalgebra

What sort of particles are described by the superalgebra? Let us again assume without loss of generality that we are in the rest frame, *i.e.* $P_\mu = (m, \vec{0})$. As for the original Poincaré group, states are labeled by m, where m^2 is the eigenvalue of P^2. For C^2 we have to do a bit of calculation:

$$\begin{aligned}
C^2 &= 2B_\mu P_\nu B^\mu P^\nu - 2B_\mu P_\nu B^\nu P^\mu \\
&\overset{RF}{=} 2m^2 B_\mu B^\mu - 2m^2 B_0^2 \\
&= 2m^2 B_k B^k,
\end{aligned}$$

and from the definition of B_μ we get:

$$\begin{aligned}
B_k &= W_k + \frac{1}{4}X_k \\
&= mS_k + \frac{1}{8}\overline{Q}\gamma_\mu\gamma^5 Q \equiv mJ_k.
\end{aligned}$$

The operator we just defined, $J_k \equiv \frac{1}{m}B_k$, is an abstraction of the ordinary spin operator, and fulfills the angular momentum algebra (just like the spin operator):

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$

and has $[J_k, Q_a] = 0$.[16] This gives us

$$C^2 = 2m^4 J_k J^k,$$

such that:

$$C^2|m, j, j_3\rangle = -m^4 j(j+1)|m, j, j_3\rangle,$$

[16] Again the proof is algebraically extensive, and again I suggest the interested reader to pursue [8].

where it can be shown that $j = 0, \frac{1}{2}, 1 \ldots$ and $j_3 = -j, -j+1, \ldots, j$ because J_k fulfils the angular momentum algebra. So, the irreducible representations of the superalgebra can be labeled by m, j, and any given set m, j will give us $2j+1$ states with different j_3.[17]

In the following we will construct all the states for a given representation with the set m, j. To do this it is very usefull to write the generators Q in terms of two-component Weyl spinors instead of four-component Dirac spinors, making explicit use of their Majorana nature, as we did in Section 2.5. We note that from the above discussion

$$[J_k, Q_A] = [J_k, \overline{Q}_{\dot{B}}] = 0.$$

We begin by claiming that for any given j_3 there must then exist a state $|\Omega\rangle$ that has the same value of j_3 and for which

$$Q_A|\Omega\rangle = 0. \tag{2.26}$$

This is called the **Clifford vacuum**.[18] To show this, start with $|\beta\rangle$, a state with j_3. Then the construction

$$|\Omega\rangle = Q_1 Q_2 |\beta\rangle$$

has these properties. First we show that (2.26) holds:

$$Q_1 Q_1 Q_2 |\beta\rangle = -Q_1 Q_1 Q_2 |\beta\rangle = 0$$

and

$$Q_2 Q_1 Q_2 |\beta\rangle = -Q_1 Q_2 Q_2 |\beta\rangle = Q_1 Q_2 Q_2 |\beta\rangle = -Q_2 Q_1 Q_2 |\beta\rangle = 0.$$

For this Clifford vacuum state we then have:

$$\begin{aligned} J_3|\Omega\rangle &= J_3 Q_1 Q_2 |\beta\rangle \\ &= Q_1 Q_2 J_3 |\beta\rangle = j_3 |\Omega\rangle, \end{aligned}$$

in other words, $|\Omega\rangle$ has the same value for j_3 as the $|\beta\rangle$ it was constructed from. We can now use the explicit expression for J_k

$$J_k = S_k - \frac{1}{4m} \overline{Q}_{\dot{B}} \overline{\sigma}_k^{\dot{B}A} Q_A,$$

in order to find the spin for this state:

$$J_k|\Omega\rangle = S_k|\Omega\rangle = j_k|\Omega\rangle,$$

meaning that $s_3 = j_3$ and $s = j$ are the eigenvalues of S_3 and S^2 for the Clifford vacuum $|\Omega\rangle$.

We can construct three more states from the Clifford vacuum:[19]

$$\overline{Q}^{\dot{1}}|\Omega\rangle, \quad \overline{Q}^{\dot{2}}|\Omega\rangle, \quad \overline{Q}^{\dot{1}}\overline{Q}^{\dot{2}}|\Omega\rangle.$$

This means that there are four possible states that can be constructed out of any state with the quantum numbers m, j, j_3. Taking a look at:

$$J_k \overline{Q}^{\dot{A}}|\Omega\rangle = \overline{Q}^{\dot{A}} J_k |\Omega\rangle = j_k \overline{Q}^{\dot{A}}|\Omega\rangle,$$

[17] Note that j is NOT the spin, but a generalization of spin.

[18] It is called the Clifford vacuum because the operators satisfy a Clifford algebra $\{Q_A, \overline{Q}_{\dot{B}}\} = 2m\sigma^0_{A\dot{B}}$. Do not confuse this with a vacuum state, it is only a name.

[19] All other possible combinations of Qs and $|\Omega\rangle$ give either one of the other four states, or the zero state which is trivial and of no interest.

this means that all these states have the same j_3 (and j) quantum numbers.[20] From the superalgebra (2.25) we have:

$$[M^{ij}, \overline{Q}^{\dot{A}}] = -(\sigma^{ij})^{\dot{A}}{}_{\dot{B}}\overline{Q}^{\dot{B}},$$

so that:

$$S_3\overline{Q}^{\dot{A}}|\Omega\rangle = \overline{Q}^{\dot{A}}S_3|\Omega\rangle - \frac{1}{2}(\overline{\sigma}_3\sigma^0)^{\dot{A}}{}_{\dot{B}}\overline{Q}^{\dot{B}}|\Omega\rangle = \left(j_3 \mp \frac{1}{2}\right)\overline{Q}^{\dot{A}}|\Omega\rangle,$$

where $-$ is for $\dot{A} = \dot{1}$ and $+$ is for $\dot{A} = \dot{2}$. We can similarly show that

$$S_3\overline{Q}^{\dot{1}}\overline{Q}^{\dot{2}}|\Omega\rangle = j_3\overline{Q}^{\dot{1}}\overline{Q}^{\dot{2}}|\Omega\rangle.$$

This means that each set of quantum numbers m, j, j_3 gives 2 states with $s_3 = j_3$, and two with $s_3 = j_3 \pm \frac{1}{2}$, giving two bosonic and two fermionic states, with the same mass.

The above explains the much repeated statement that any supersymmetry theory has an equal number of bosons and fermions, which, incidentally, is not true.

Theorem: For any representation of the superalgebra where P_μ is a one-to-one operator there is an equal number of boson and fermion states.

To show this, divide the representation into two sets of states, one with bosons and one with fermions. Let $\{Q_A, \overline{Q}_{\dot{B}}\}$ act on the members of the set of bosons. $\overline{Q}_{\dot{B}}$ transforms bosons to fermions and Q_A does the reverse mapping. If P_μ is one-to-one, then so is $\{Q_A, \overline{Q}_{\dot{B}}\} = 2\sigma^\mu{}_{A\dot{B}}P_\mu$. Thus there must be an equal number in both sets.[21]

Let us expand on the two simplest examples. For $j = 0$ the Clifford vacuum $|\Omega\rangle$ has $s = 0$ and is a bosonic state. There are two states $\overline{Q}^{\dot{A}}|\Omega\rangle$ with $s = \frac{1}{2}$ and $s_3 = \mp\frac{1}{2}$ and one state $\overline{Q}^{\dot{1}}\overline{Q}^{\dot{2}}|\Omega\rangle$ with $s = 0$ and $s_3 = 0$. In total there are two scalar particles and two spin-$\frac{1}{2}$ fermions. Note that all these particles have the same mass. We will later refer to this set of states as the **scalar superfield**.

For $j = \frac{1}{2}$ we have two Clifford vacua $|\Omega\rangle$ with $j_3 = \pm\frac{1}{2}$, and with $s = \frac{1}{2}$ and $s_3 = \pm\frac{1}{2}$ (thus they are fermionic states). For the moment we label them as $|\Omega; \frac{1}{2}\rangle$ and $|\Omega; -\frac{1}{2}\rangle$. From each of these we can construct two further fermion states $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega; \pm\frac{1}{2}\rangle$ with $s_3 = \mp\frac{1}{2}$. In addition to this we have the states $\bar{Q}^{\dot{1}}|\Omega; \frac{1}{2}\rangle$ and $\bar{Q}^{\dot{2}}|\Omega; -\frac{1}{2}\rangle$ with $s_3 = 0$, the state $\bar{Q}^{\dot{2}}|\Omega; \frac{1}{2}\rangle$ with $s_3 = 1$, and the state $\bar{Q}^{\dot{1}}|\Omega; -\frac{1}{2}\rangle$ has $s_3 = -1$. Together these states can form two fermions with $s = \frac{1}{2}$ and $s_3 = \pm\frac{1}{2}$, one massive vector particle with $s = 1$, and $s_3 = 1, 0, -1$, and one scalar with $s = 0$.[22] We will later refer to this set of states as the **vector superfield**.

Exercise: What are the states for $j = 1$?

We should use the term particle here very lightly since the states we have found are spinor states. A real Dirac fermion can only be described by a $j = 0$ representation and a

[20] The same can easily be shown for $\overline{Q}^{\dot{1}}\overline{Q}^{\dot{2}}|\Omega\rangle$.

[21] Observe that this tells us that there must be an equal number of states in both sets, not particles.

[22] For massless particles, $m = 0$, we can form a vector particle with $s_3 = \pm 1$ and one extra scalar.

complex conjugate representation, thus having four degrees of freedom (d.o.f.). In field theory calculations, when the fermion is on-shell, two of these are eliminated in the Dirac equation, thus we get the expected two d.o.f. for a fermion.

Chapter 3

Superspace

In this chapter we will introduce a very handy notation for considering supersymmetry transformations effected by the superalgebra, or, more correctly, the elements of the super-Poincaré group. This is called superspace. In order to do this we first need to know a little about the properties of Grassman numbers.

3.1 Superspace calculus

Grassman numbers θ are numbers that anti-commute with each others but not with ordinary numbers. We will here use four such numbers, and in addition we want to place them in Weyl spinors:[1]

$$\{\theta^A, \theta^B\} = \{\theta^A, \overline{\theta}^{\dot{B}}\} = \{\overline{\theta}^{\dot{A}}, \theta^B\} = \{\overline{\theta}^{\dot{A}}, \overline{\theta}^{\dot{B}}\} = 0.$$

From this we get the relationships:[2]

$$\begin{aligned} \theta_A^2 &= \theta_A\theta_A = -\theta_A\theta_A = 0, && (3.1)\\ \theta^2 &\equiv \theta\theta \equiv \theta^A\theta_A = -2\theta_1\theta_2, && (3.2)\\ \bar{\theta}^2 &\equiv \bar{\theta}\bar{\theta} \equiv \bar{\theta}_{\dot{A}}\bar{\theta}^{\dot{A}} = 2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}}. && (3.3) \end{aligned}$$

Notice that if we have a function f of a Grassman number, say θ_A, then the all-order expansion of that function in terms of θ_A, is

$$f(\theta_A) = a_0 + a_1\theta_A, \tag{3.4}$$

there simply are no more terms because of (3.1).

We now need to define differentiation and integration on these numbers in order to create a calculus for them.

[1]We can already see how this can be handy: if we consistently use $\theta^A Q_A$ and $\bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}}$ instead of only Q_A and $\bar{Q}^{\dot{A}}$ in Eqs. (2.22)–(2.25) we can actually rewrite the superalgebra as an ordinary Lie algebra because of these commutation properties.

[2]There is no summation implied in the first line.

Definition: We define differentiation by:[a]

$$\partial_A \theta^B \equiv \frac{\partial}{\partial \theta^A} \theta^B \equiv \delta_A{}^B,$$

with a product rule

$$\begin{aligned} \partial_A(\theta^{B_1}\theta^{B_2}\theta^{B_3}\dots\theta^{B_n}) \equiv & (\partial_A\theta^{B_1})\theta^{B_2}\theta^{B_3}\dots\theta^{B_n} \\ & -\theta^{B_1}(\partial_A\theta^{B_2})\theta^{B_3}\dots\theta^{B_n} \\ & +\dots+(-1)^{n-1}\theta^{B_1}\theta^{B_2}\dots(\partial_A\theta^{B_n}). \end{aligned} \tag{3.5}$$

[a]Note that this has no infinitesimal interpretation.

Definition: We define integration by $\int d\theta_A \equiv 0$ and $\int d\theta_A \theta_A \equiv 1$ and we demand linearety:

$$\int d\theta_A [af(\theta_A) + bg(\theta_A)] \equiv a \int d\theta_A f(\theta_A) + b \int d\theta_A g(\theta_A).$$

This has one surprising property. If we take the integral of (3.4) we get:

$$\int d\theta_A f(\theta_A) = a_1 = \partial^A f(\theta_A),$$

meaning that differentiation and integration has the same effect on Grassman numbers.

To integrate over multiple Grassman numbers we define volume elements for the Weyl spinors

Definition:

$$d^2\theta \equiv -\frac{1}{4} d\theta^A d\theta_A,$$

$$d^2\overline{\theta} \equiv -\frac{1}{4} d\overline{\theta}_{\dot{A}} d\overline{\theta}^{\dot{A}},$$

$$d^4\theta \equiv d^2\theta d^2\overline{\theta}.$$

This means that

$$\int d^2\theta \ \theta\theta = 1$$

$$\int d^2\overline{\theta} \ \overline{\theta\theta} = 1$$

$$\int d^4\theta \ (\theta\theta)(\overline{\theta\theta}) = 1$$

Delta functions of Grassmann variables are given by:

$$\delta(\theta_A) = \theta_A$$
$$\delta^2(\theta_A) = \theta\theta$$
$$\delta^2(\overline{\theta}^{\dot{A}}) = \overline{\theta\theta}$$

and these functions satisfy (just as the usual definition of delta functions):

$$\int d\theta_A f(\theta_A)\delta(\theta_A) = f(0).$$

3.2 Superspace definition (Salam & Strathdee [10])

Superspace is a coordinate system where supersymmetry transformations are manifest, in other words, the action of elements in the super-Poincaré group (SP) based on the superalgebra are treated like Lorentz-transformations are in Minkowski space.

Definition: **Superspace** is an eight-dimension manifold that can be constructed from the **coset space** of the super-Poincaré group (SP) and the Lorentz group (L), SP/L, by giving coordinates $z^\pi = (x^\mu, \theta^A, \bar{\theta}^{\dot{A}})$, where x^μ are the ordinary Minkowski coordinates, and where θ_A and $\bar{\theta}^{\dot{A}}$ are four Grassman (anti-commuting) numbers, being the parameters of the Q-operators in the algebra.

To see this we begin by writing a general element of SP, $g \in SP$, as[3]

$$g = \exp[-ix^\mu P_\mu + i\theta^A Q_A + i\overline{\theta}_{\dot{A}}\overline{Q}^{\dot{A}} - \frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}],$$

where x^μ, θ^A, $\bar{\theta}_{\dot{A}}$ and $\omega_{\rho\nu}$ constitute the parametrization of the group, and P_μ, Q_A, $\bar{Q}^{\dot{A}}$ and $M_{\rho\nu}$ are the generators. We can now find members in SP/L simply by setting $\omega_{\mu\nu} = 0$.[4] The remaining parameters of SP/L then span superspace.

As we are physicists we also want to know the dimensions of our new parameters. To do this we first look at Eq. (2.23):

$$\{Q_A, \overline{Q}_{\dot{B}}\} = 2\sigma^\mu{}_{A\dot{B}}P_\mu$$

we know that P_μ has mass dimension $[P_\mu] = M$. This means that $[Q^2] = M$ and $[Q] = M^{\frac{1}{2}}$. In the exponential, all terms must have mass dimension zero to make sense. This means that $[\theta Q] = 0$, and therefore $[\theta] = M^{-\frac{1}{2}}$.

In order to show the effect of supersymmetry transformations, we begin by noting that any SP transformation can effectively be written in the following way:

$$L(a, \alpha) = \exp[-ia^\mu P_\mu + i\alpha^A Q_A + i\bar{\alpha}^{\dot{A}}\bar{Q}_{\dot{A}}],$$

[3]We hava already used this property, but this is what is formally called an **exponential map** of the Lie algebra to the Lie group. For matrix Lie groups this is simply the matrix exponential shown here. Technicaly this provides a **local cover** of the group around small values for the parameters.

[4]Given that we can prove that SP/L is indeed a coset group, which follows from our earlier definition of coset groups and that L is a normal subgroup of SP.

because one can show that[5]

$$\exp\left[-\frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right]L(a,\alpha) = L(\Lambda a, S(\Lambda)\alpha)\exp\left[-\frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right], \tag{3.6}$$

i.e. all that a Lorentz boost does is to transform spacetime coordinates by $\Lambda(M)$ and Weyl spinors by $S(\Lambda(M))$, which is a spinor representation of $\Lambda(M)$. Thus, we can pick frames, do our thing with the transformation, and boost back to any frame we wanted. In addition, since P_μ commutes with all the Qs, when we speak of the supersymmetry transformation we usually mean just the transformation

$$\delta_S = \alpha^A Q_A + \overline{\alpha}_{\dot{A}}\overline{Q}^{\dot{A}}. \tag{3.7}$$

We can now find the transformation of superspace coordinates under a supersymmetry transformation, just as we have all seen the transformation of Minkowski coordinates under Lorentz transformations. The effect of $g_0 = L(a,\alpha)$ on a superspace coordinate $z^\pi = (x^\mu, \theta^A, \bar{\theta}_{\dot{A}})$ is defined by the mapping $z^\pi \to z'^\pi$ given by $g_0 e^{iz^\pi K_\pi} = e^{iz'^\pi K_\pi}$ where $K_\pi = (P_\mu, Q_A, \bar{Q}^{\dot{A}})$ are the generators of the coset group SP/L. We have[6]

$$\begin{aligned} g_0 e^{iz^\pi K_\pi} &= \exp(-ia^\nu P_\nu + i\alpha^B Q_B + i\overline{\alpha}_{\dot{B}}\overline{Q}^{\dot{B}})\exp(iz^\pi K_\pi) \\ &= \exp(-ia^\nu P_\nu + i\alpha^B Q_B + i\overline{\alpha}_{\dot{B}}\overline{Q}^{\dot{B}} + iz^\pi K_\pi \\ &\quad -\frac{1}{2}[-ia^\nu P_\nu + i\alpha^B Q_B + i\overline{\alpha}_{\dot{B}}\overline{Q}^{\dot{B}}, iz^\pi K_\pi] + \ldots) \end{aligned}$$

Here we take a closer look at the commutator:[7]

$$\begin{aligned} [\ ,\ ] &= [\alpha^B Q_B, \overline{\theta}_{\dot{A}}\overline{Q}^{\dot{A}}] + [\overline{\alpha}_{\dot{B}}\overline{Q}^{\dot{B}}, \theta^A Q_A] \\ &= -\alpha^B\overline{\theta}_{\dot{A}}\epsilon^{\dot{A}\dot{C}}\{Q_B, \overline{Q}_{\dot{C}}\} - \overline{\alpha}_{\dot{B}}\theta^A\epsilon^{\dot{B}\dot{C}}\{\overline{Q}_{\dot{C}}, Q_A\} \\ &= -2\alpha^B\overline{\theta}_{\dot{A}}\epsilon^{\dot{A}\dot{C}}\sigma^\mu{}_{B\dot{C}}P_\mu - \overline{\alpha}_{\dot{B}}\theta^A\epsilon^{\dot{B}\dot{C}}\sigma^\mu{}_{A\dot{C}}P_\mu \\ &= (-2\alpha^B\overline{\theta}^{\dot{C}}\sigma^\mu{}_{B\dot{C}} - 2\overline{\alpha}^{\dot{C}}\theta^A\sigma^\mu{}_{A\dot{C}})P_\mu \end{aligned}$$

We can relabel $B = A$ and $\dot{C} = \dot{A}$ which leads to

$$-\frac{1}{2}[\ ,\ ] = (\alpha^A\sigma^\mu{}_{A\dot{A}}\overline{\theta}^{\dot{A}} - \theta^A\sigma^\mu{}_{A\dot{A}}\overline{\alpha}^{\dot{A}})P_\mu.$$

The commutator is proportional with P_μ, and will therefore commute with all operators, in particular the higher terms in the Campbell-Baker-Hausdorff expansion, meaning that the series reduces to

$$\begin{aligned} & g_0 e^{iZ^\pi K_\pi} \\ = \ & \exp[i(-x^\mu - a^\mu + i\alpha^A\sigma^\mu{}_{A\dot{A}}\overline{\theta}^{\dot{A}} - i\theta^A\sigma^\mu{}_{A\dot{A}}\overline{\alpha}^{\dot{A}})P_\mu + i(\theta^A + \alpha^A)Q_A + i(\overline{\theta}_{\dot{A}} + \overline{\alpha}_{\dot{A}})\overline{Q}^{\dot{A}}]. \end{aligned}$$

[5]Fortunately we are not going to do this because it is messy, but it can be done using the algebra of the group and the series expansion of the exponential function. Note, however, that the proof rests on the Ps and Qs forming a closed set, which we saw in the algebra Eqs. (2.22)–(2.25).

[6]Here we use Campbell-Baker-Hausdorff expansion $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}-\frac{1}{2}[\hat{A},\hat{B}]+\cdots}$ where the next term contains commutators of the first commutator and the operators $\hat{A}$ and $\hat{B}$.

[7]Using that P_μ commutes with all elements in the algebra, as well as $[\theta^A Q_A, \xi^B Q_B] = \theta^A\xi^B\{Q_A, Q_B\} = 0$, and the same for $\bar{Q}^{\dot{B}}$.

So superspace coordinates transform under supersymmetry transformations as:

$$(x^\mu, \theta^A, \overline{\theta}_{\dot{A}}) \to f(a^\mu, \alpha^A, \overline{\alpha}_{\dot{A}}) = (x^\mu + a^\mu - i\alpha^A \sigma^\mu{}_{A\dot{A}} \overline{\theta}^{\dot{A}} + i\theta^A \sigma^\mu{}_{A\dot{A}} \overline{\alpha}^{\dot{A}}, \theta^A + \alpha^A, \overline{\theta}_{\dot{A}} + \overline{\alpha}_{\dot{A}}). \quad (3.8)$$

As a by-product we can now write down a differential representation for the supersymmetry generators by applying the standard expression for the generators X_i of a Lie algebra, given the functions f_π for the transformation of the parameters:

$$X_j = \frac{\partial f_\pi}{\partial a_j} \frac{\partial}{\partial z_\pi}$$

which gives us:[8]

$$\begin{aligned} P_\mu &= i\partial_\mu && (3.9) \\ iQ_A &= -i(\sigma^\mu \overline{\theta})_A \partial_\mu + \partial_A && (3.10) \\ i\overline{Q}^{\dot{A}} &= -i(\overline{\sigma}^\mu \theta)^{\dot{A}} \partial_\mu + \partial^{\dot{A}} && (3.11) \end{aligned}$$

Exercise: Check that Eqs. (3.9)–(3.11) fulfil the superalgebra in Eqs. (2.22)–(2.24).

3.3 Covariant derivatives

Similar to the properties of covariant derivatives for gauge transformations in gauge theories, it would be nice to have a derivative that is invariant under supersymmetry transformations, *i.e.* commutes with supersymmetry operators. Obviously $P_\mu = i\partial_\mu$ does this, but more general covariant derivatives can be made.

Definition: The following covariant derivatives commute with supersymmetry transformations:

$$\begin{aligned} D_A &\equiv \partial_A + i(\sigma^\mu \overline{\theta})_A \partial_\mu, && (3.12) \\ \bar{D}_{\dot{A}} &\equiv -\partial_{\dot{A}} - i(\theta\sigma^\mu)_{\dot{A}} \partial_\mu. && (3.13) \end{aligned}$$

These can be shown to satisfy relations that are useful in calculations:

$$\begin{aligned} \{D_A, D_B\} &= \{\overline{D}_{\dot{A}}, \overline{D}_{\dot{B}}\} = 0 && (3.14) \\ \{D_A, \overline{D}_{\dot{B}}\} &= -2\sigma^\mu_{A\dot{B}} P_\mu && (3.15) \\ D^3 = \overline{D}^3 &= 0 && (3.16) \\ D^A \overline{D}^2 D_A &= \overline{D}_{\dot{A}} D^2 \overline{D}^{\dot{A}} && (3.17) \end{aligned}$$

From the covariant derivatives we can construct projection operators.

[8]We define the generators X_i as $-iP_\mu$, iQ_A and iQ_B respectively.

Definition: The operators

$$\pi_+ \equiv -\frac{1}{16\Box}\overline{D}^2 D^2, \quad (3.18)$$

$$\pi_- \equiv -\frac{1}{16\Box}D^2\overline{D}^2, \quad (3.19)$$

$$\pi_T \equiv \frac{1}{8\Box}\overline{D}_{\dot{A}}D^2\overline{D}^{\dot{A}}, \quad (3.20)$$

with $\Box \equiv \partial_\mu\partial^\mu$, are projection operators, *i.e.* they fulfill:

$$\pi^2_{\pm,T} = \pi_{\pm,T} \quad (3.21)$$

$$\pi_+\pi_- = \pi_+\pi_T = \pi_-\pi_T = 0 \quad (3.22)$$

$$1 = \pi_+ + \pi_- + \pi_T \quad (3.23)$$

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