



UiO : **University of Oslo**

Uke 9: Diskret Fourier Transform, I

Jo Inge Buskenes

Institutt for informatikk, Universitetet i Oslo

INF3470/4470, høst 2012



Dagens temaer

Sampling og periodisitet

DFT

DFT og DTFT

Tema

Sampling og periodisitet

Sampling og periodisitet

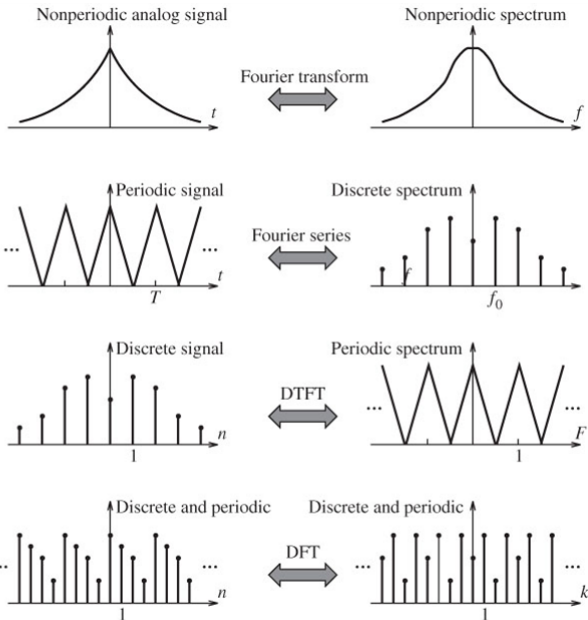
- ▶ Sampling i tid \implies periodisitet i frekvens
- ▶ Sampling i frekvens \implies periodisitet i tid
- ▶ 4 muligheter:

Tid	Frekvens	Metode
kontinuerlig, aperiodisk	\leftrightarrow kontinuertlig, aperiodisk	FT
kontinuerlig, periodisk	\leftrightarrow diskret, aperiodisk	Fourier rekker
diskret, aperiodisk	\leftrightarrow kontinuertlig, periodisk	DTFT
diskret, periodisk	\leftrightarrow diskret, periodisk	DFS/DFT

FIGURE 8.1

Features of the various transforms. A nonperiodic analog signal has a nonperiodic analog spectrum described by its Fourier transform.

Sampling in one domain induces periodicity in the other. Thus, a periodic analog signal has a discrete spectrum (described by its Fourier series) and a discrete nonperiodic signal has a periodic spectrum (described by the DTFT). If a signal is periodic and discrete, its spectrum is also periodic and discrete (and described by the DFT)



Tema

DFT

Definisjoner

Egenskaper

Noen DFT-par

DFT

Definisjon

$$\blacktriangleright X_{\text{DFT}}[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}, \quad k = 0, 1, 2, \dots, N-1$$

$$\blacktriangleright x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DFT}}[k] e^{j2\pi \frac{k}{N} n}, \quad n = 0, 1, 2, \dots, N-1$$

Notasjon

$$\blacktriangleright \text{Notasjon: } x[n] \xleftrightarrow[N]{\text{DFT}} X_{\text{DFT}}[k] \quad (= X[k])$$

Merk:

- ▶ Diskret tid/frekvens \rightarrow diskrete summer.
- ▶ Periodisk tid/frekvens \rightarrow evaluerer **en** periode.
- ▶ Dette kan en datamaskin håndtere!

Properties of the DFT

The properties of the DFT are summarized in Table 8.3. They are strikingly similar to other frequency-domain transforms, but must always be used in keeping with implied periodicity (of the DFT and IDFT) in both domains.

Property	Signal	DFT	Remarks
Shift	$x[n - n_0]$	$X_{\text{DFT}}[k]e^{-j2\pi kn_0/N}$	No change in magnitude
Shift	$x[n - 0.5N]$	$(-1)^k X_{\text{DFT}}[k]$	Half-period shift for even N
Modulation	$x[n]e^{j2\pi nk_0/N}$	$X_{\text{DFT}}[k - k_0]$	
Modulation	$(-1)^n x[n]$	$X_{\text{DFT}}[k - 0.5N]$	Half-period shift for even N
Reversal	$x[-n]$	$X_{\text{DFT}}[-k]$	This is <i>circular</i> flipping
Product	$x[n]y[n]$	$\frac{1}{N} X_{\text{DFT}}[k] \otimes Y_{\text{DFT}}[k]$	The convolution is <i>periodic</i>
Convolution	$x[n] \otimes y[n]$	$X_{\text{DFT}}[k] Y_{\text{DFT}}[k]$	The convolution is <i>periodic</i>
Correlation	$x[n] \otimes^* y[n]$	$X_{\text{DFT}}[k] Y_{\text{DFT}}^*[k]$	The correlation is <i>periodic</i>
Central ordinates	$x[0] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DFT}}[k] \quad X_{\text{DFT}}[0] = \sum_{n=0}^{N-1} x[n]$		
Central ordinates	$x[\frac{N}{2}] = \frac{1}{N} \sum_{k=0}^{N-1} (-1)^k X_{\text{DFT}}[k] \quad (N \text{ even}) \quad X_{\text{DFT}}[\frac{N}{2}] = \sum_{n=0}^{N-1} (-1)^n x[n] \quad (N \text{ even})$		
Parseval's relation	$\sum_{n=0}^{N-1} x[n] ^2 = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DFT}}[k] ^2$		

Egenskaper til DFT-transformasjonen

- ▶ Linearitet:

$$\mathcal{F}_{\text{DFT}}\{a_1x_1[n] + a_2x_2[n]\} = a_1X_1[k] + a_2X_2[k]$$

- ▶ Energibevaring (Parsevals relasjon):

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_{\text{DFT}}[k]|^2$$

Skifting i tid

- ▶ Å skifte noe i tid tilsvarer å legge på et faseskift i frekvens.

$$\begin{aligned}
 \mathcal{F}_{\text{DFT}}\{x[n - n_0]\} &= \sum_{n=0}^{N-1} x[n - n_0] e^{-j2\pi \frac{k}{N}n} && m = n - n_0 \\
 &= \sum_{m=n_0}^{n_0+N-1} x[m] e^{-j2\pi \frac{k}{N}(m+n_0)} \\
 &= e^{-j2\pi \frac{k}{N}n_0} \sum_{m=n_0}^{n_0+N-1} x[m] e^{-j2\pi \frac{k}{N}m} \\
 &= e^{-j2\pi \frac{k}{N}n_0} X[k]
 \end{aligned}$$

- ▶ Vi kan erstatte $\sum_{m=n_0}^{n_0+N-1}$ med $\sum_{m=0}^{N-1}$ siden $x[m] e^{-j2\pi \frac{k}{N}m}$ er periodisk med periodisitet N , ergo summeres samme samplene i begge tilfeller.

Modulering

- ▶ Å legge på et faseskift i tid tilsvarer å skifte i frekvens.

$$\begin{aligned}\mathcal{F}_{\text{DFT}}\{x[n] e^{j2\pi \frac{k_0}{N} n}\} &= \sum_{n=0}^{N-1} x[n] e^{j2\pi \frac{k_0}{N} n} e^{-j2\pi \frac{k}{N} n} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k-k_0}{N} n} \\ &= X[k - k_0]\end{aligned}$$

- ▶ Å skifte noe i ett domene gir et faseskift i det andre.

Reversering

- ▶ Hvis et signal er odde- eller likesymmetrisk er det enkelt å finne transformen til det reverserte signalet.
- ▶ Hvis likesymmetri, $x[n] = x[-n]$:

$$\begin{aligned} \mathcal{F}_{\text{DFT}}\{x[-n]\} &= \sum_{n=0}^{N-1} x[-n] e^{-j2\pi \frac{k}{N}n} & m = -n \\ &= \sum_{m=0}^{N-1} x[m] e^{-j2\pi \frac{-k}{N}m} = X[-k] \end{aligned}$$

- ▶ Hvis odde $x[n] = -x[-n]$:

$$\mathcal{F}_{\text{DFT}}\{-x[-n]\} = -X[-k]$$

- ▶ Reversering i tid gir “negative” frekvenser.

Konjugert symmetri

$$\begin{aligned}
 \mathcal{F}_{\text{DFT}}\{X^*[n]\} &= \sum_{n=-\infty}^{\infty} x^*[n] e^{-j2\pi \frac{k}{N}n} \\
 &= \left(\sum_{n=-\infty}^{\infty} x[n] (e^{-j2\pi \frac{k}{N}n})^* \right)^* \\
 &= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi \frac{-k}{N}n} \right)^* \\
 &= X^*[-k]
 \end{aligned}$$

- ▶ Hvis $x[n]$ er reell, så er $x[n] = x^*[n]$ og også $X[k] = X^*[-k]$.
- ▶ Kombinerer vi disse symmetri-egenskapene med det at DFT'en er periodisk om N ser vi at vi kun trenger regne på frekvenser opp til halve samplingsfrekvensen.

Sirkulær/periodisk konvolusjon

$$\begin{aligned}
 \mathcal{F}_{\text{DFT}}\{x[n] \circledast h[n]\} &= \sum_{n=0}^{N-1} (x[n] \circledast h[n]) e^{-j2\pi \frac{k}{N}n} \\
 &= \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1} x[n-k] h[k] \right) e^{-j2\pi \frac{k}{N}n} \\
 &= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x[n-k] h[k] e^{-j2\pi \frac{k}{N}n} \\
 &= \sum_{k=0}^{N-1} h[k] \sum_{n=0}^{N-1} x[n-k] e^{-j2\pi \frac{k}{N}n} && m = n - k \\
 &= \sum_{k=0}^{N-1} h[k] \sum_{m=0}^{N-1} x[m] e^{-j2\pi \frac{k}{N}(m+k)} \\
 &= \sum_{k=0}^{N-1} h[k] e^{-j2\pi \frac{k}{N}k} \sum_{m=0}^{N-1} x[m] e^{-j2\pi \frac{k}{N}m} \\
 &= X_2[k] X_1[k]
 \end{aligned}$$

Finne $x[0]$ eller $X_{\text{DFT}}[0]$

$$x[0] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DFT}}[k] e^{j2\pi \frac{k}{N} n} \Bigg|_{n=0} = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DFT}}[k]$$

$$X_{\text{DFT}}[0] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n} \Bigg|_{k=0} = \sum_{n=0}^{N-1} x[n] \quad \text{DC verdien}$$

DFT par

8.3.8 Some Useful DFT Pairs

The DFT of finite sequences defined mathematically often results in very unwieldy expressions and explains the lack of many “standard” DFT pairs. However, the following DFT pairs are quite useful and easy to obtain from the defining relation and properties:

$$\downarrow \{1, 0, 0, \dots, 0\}(\text{impulse}) \leftarrow \boxed{\text{DFT}} \Rightarrow \downarrow \{1, 1, 1, \dots, 1\} \quad (\text{constant}) \quad (8.7)$$

$$\downarrow \{1, 1, 1, \dots, 1\}(\text{constant}) \leftarrow \boxed{\text{DFT}} \Rightarrow \downarrow \{N, 0, 0, \dots, 0\} \quad (\text{impulse}) \quad (8.8)$$

$$\alpha^n(\text{exponential}) \leftarrow \boxed{\text{DFT}} \Rightarrow \frac{1 - \alpha^N}{1 - \alpha e^{-j2\pi k/N}} \quad (8.9)$$

$$\cos\left(2\pi n \frac{k_0}{N}\right)(\text{sinusoid}) \leftarrow \boxed{\text{DFT}} \Rightarrow 0.5N\delta[k - k_0] + 0.5N\delta[k - (N - k_0)] \quad (\text{impulse pair}) \quad (8.10)$$

Tema

DFT og DTFT

DFT av periodiske signaler; DFS

DFT av ikke-periodiske signaler

Sammenheng DFT og DTFT, frekvens og tid

$$X_{\text{DFT}}[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DFT}}[k] e^{j2\pi \frac{k}{N} n}$$

$$X_{\text{DTFT}}(F) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi \mathbf{F} n} \quad x[n] = \frac{1}{N} \int_{-1/2}^{1/2} X_{\text{DTFT}}(F) e^{j2\pi \mathbf{F} n} dF$$

- ▶ Substituerer vi $F = \frac{k}{N}$ og lar $N \rightarrow \infty$ burde altså DFT være en god approximasjon til DTFT.

$$\begin{aligned} X_{\text{DFT}}[k] &= X(\Omega) \Big|_{\Omega = \frac{2\pi k}{N}}, \quad x = 0, 1, \dots, N-1 \\ &= X(z) \Big|_{z = e^{j2\pi k/N}}, \quad x = 0, 1, \dots, N-1 \end{aligned}$$

Sammenheng DFT og DTFT, frekvens og tid

Når er DFT og DTFT like?

$$\begin{aligned}
 y[n] &= \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j2\pi \frac{k}{N} n} \quad \text{hvor} \quad Y(k) = X(F) \Big|_{F=\frac{k}{N}} = \sum_{l=-\infty}^{\infty} x[l] e^{-j2\pi \frac{k}{N} l} \\
 &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{l=-\infty}^{\infty} x[l] e^{-j2\pi \frac{k}{N} l} \right) e^{j2\pi \frac{k}{N} n} \\
 &= \sum_{l=-\infty}^{\infty} x[l] \left(\frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi \frac{k}{N} (n-l)} \right) \xrightarrow{\text{dette blir}} \begin{cases} 1 & \text{når } l = n + mN, \quad m \in \mathbb{Z} \\ 0 & \text{ellers} \end{cases} \\
 &= \sum_{m=-\infty}^{\infty} x[n + mN]
 \end{aligned}$$

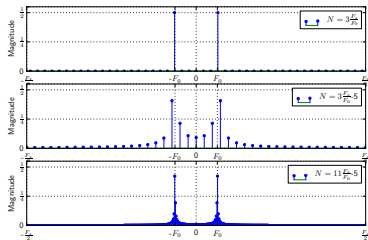
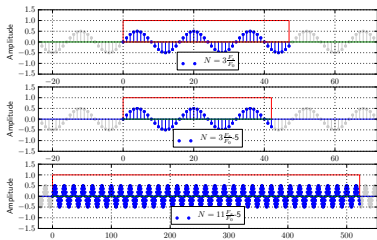
Sammenheng DFT og DTFT, frekvens og tid

▶ Konklusjoner?

- ▶ DTFT'en vil være *eksakt lik* DFT'en hvis vi setter $F = \frac{k}{N}$ og
 - ▶ $x[n]$ er en endelig diskret sekvens med lengde N , eller
 - ▶ $x[n]$ diskret periodisk sekvens med periodisitet N . Da er $DFT(1/N X_{DFT})$ eksakt match.
- ▶ Gjelder ikke dette vil DFT kun være en approximasjon til DTFT.

Endelig lengde data

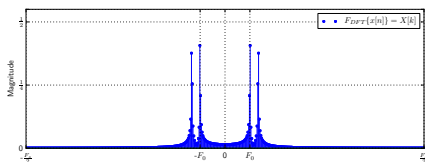
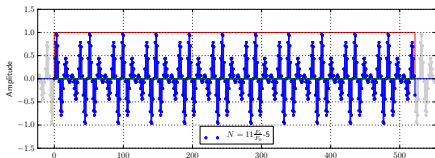
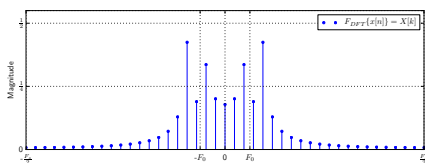
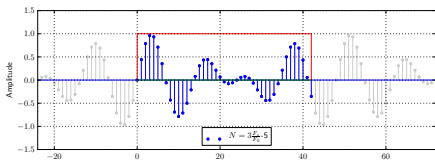
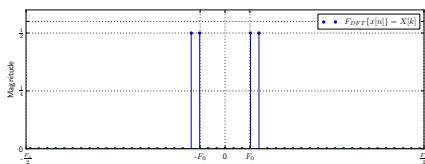
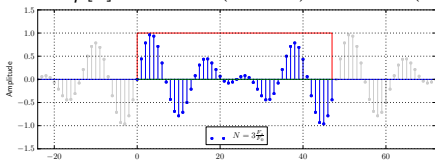
Forskjellige utsnitt fra det periodiske signalet $x_p[n] = 0.5 \sin(2\pi F_0 n)$:



Endelig lengde data

Forskjellige utsnitt fra det periodiske signalet

$$x_p[n] = 0.5 \sin(2\pi F_0 n) + 0.5 \sin(2\pi F_1 n):$$



DFT av periodiske signaler

$$\blacktriangleright X_{DFS}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}, \quad k = 0, 1, \dots, N-1$$

diskret Fourier serie

$$\blacktriangleright x[n] = \sum_{k=0}^{N-1} X_{DFS}[k] e^{j2\pi \frac{k}{N} n}, \quad n = 0, 1, \dots, N-1$$

invers diskret Fourier serie

- ▶ Identisk med DFT med unntak av skaleringsfaktor!
- ▶ Likheter, DFT ($X_{DFT}[k]$) av samplet periodisk signal $x(t)$ og dets DFS ($X[k]$):
 - ▶ Hvis $x(t)$ er båndbegrenset og samplet et heltallig antall perioder, er $X_{DFT}[k] = NX[k]$.
 - ▶ Hvis $x(t)$ ikke er båndbegrenset, så er det **aliasing**.
 - ▶ Hvis $x(t)$ ikke er samplet et heltallig antall periodier, så er det **lekkasje**.

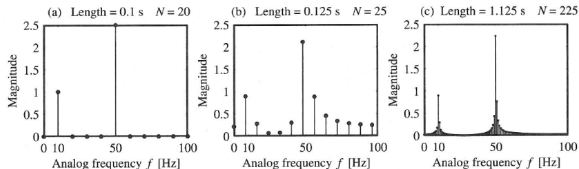
Example 8.11

The Effects of Leakage

The signal $x(t) = 2 \cos(20\pi t) + 5 \cos(100\pi t)$ is sampled at intervals of $t_s = 0.005$ s for three different durations, 0.1 s, 0.125 s, and 1.125 s. Explain the DFT spectrum for each duration.

The sampling frequency is $S = 1/t_s = 200$ Hz. The frequencies in $x(t)$ are $f_1 = 10$ Hz and $f_2 = 50$ Hz. The two-sided spectrum of $x(t)$ will show a magnitude of 1 at 10 Hz and 2.5 at 50 Hz. The fundamental frequency is 10 Hz, and the common period of $x(t)$ is 0.1 s. We have the following results with reference to Figure E8.11, which shows the DFS magnitude ($|X_{\text{DFS}}|/N$) up to the folding index (or 100 Hz).

FIGURE E.8.11 DFT results for Example 8.11



- (a) The duration of 0.1 s corresponds to one full period, and $N = 20$. No leakage is present, and the DFS results reveal an *exact match* to the spectrum of $x(t)$. The nonzero components appear at the integer indices $k_1 = NF = Nf_1/S = 1$ and $k_2 = NF_2 = Nf_2/S = 5$ (corresponding to 10 Hz and 50 Hz, respectively).
- (b) The duration of 0.125 s corresponds to 1.25 periods. So, leakage is present. The number of samples over 0.125 s is $N = 25$. The largest components in the 25-point DFS appear at the integer indices closest to $k_1 = NF = Nf_1/S = 1.25$ (i.e., $k = 1$ or 8 Hz) and $k_2 = Nf_2/S = 6.25$ (i.e., $k = 6$ or 48 Hz).
- (c) The duration of 1.125 s corresponds to 11.25 periods. So, leakage is present. The number of samples over 1.125 s is $N = 225$. The largest components in the 225-point DFS appear at the integer indices closest to $k_1 = NF = Nf_1/S = 11.25$ (i.e., $k = 11$ or 9.78 Hz) and $k_2 = Nf_2/S = 56.25$ (i.e., $k = 56$ or 49.78 Hz).

Comment: The last two spectra show leakage. The spectrum of the signal sampled for the longer duration (1.125 s) produces the smaller leakage.

DFT av ikke-periodiske signaler

- ▶ Må anta at signalet er endelig periodisk.
- ▶ For å sample det må vi bestemme antall sampler N , og samplingsrate S .
 - ▶ Siden antatt endelig, vil $f_{max} = \infty$; aliasing.
 - ▶ Velg $S = 2B$ der B er større enn største signifikante frekvens.
 - ▶ Oppløsningen spektralt gitt fra S/N .
 - ▶ Øke oppløsning ved å null-utvide. Gir ikke mer informasjon!
 - ▶ Mer info fås fra større N (sample for en lengere tid).