Introduction to Robotics (Fag 3480)

Vår 2010

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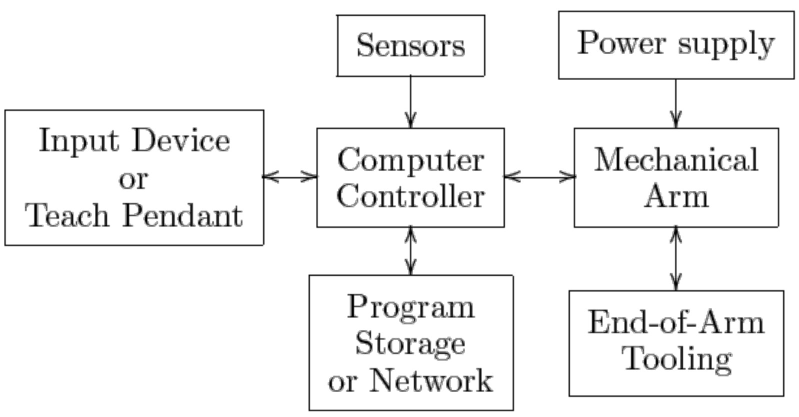
Ch. 2: Rigid Body Motions and Homogeneous Transforms

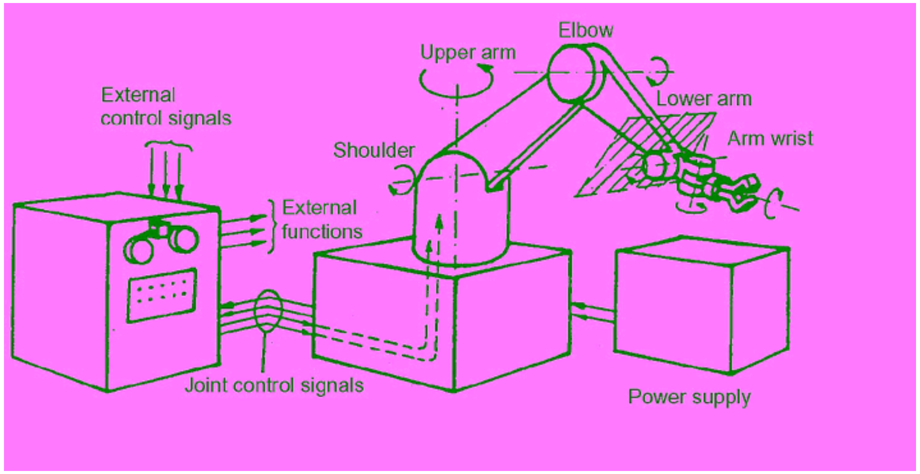
Industrial robots

High precision and repetitive tasks

Pick and place, painting, etc

Hazardous environments

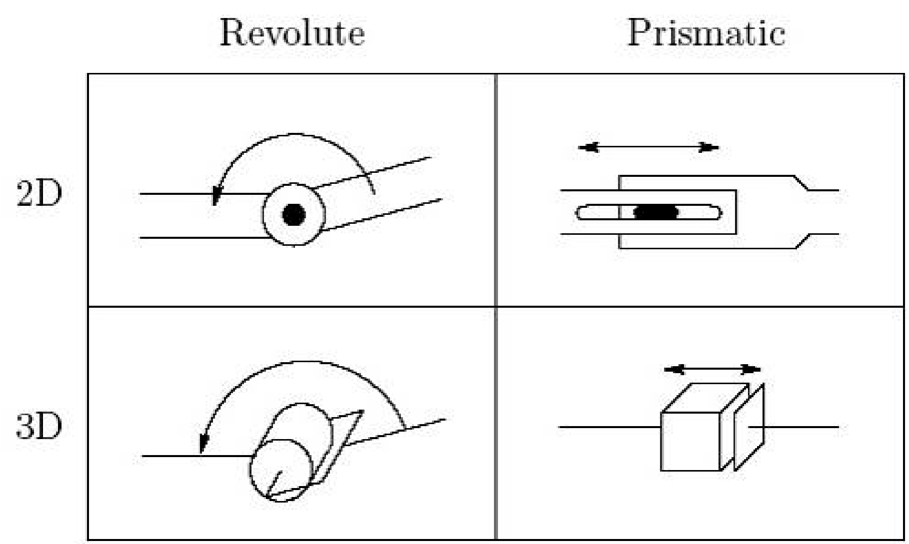


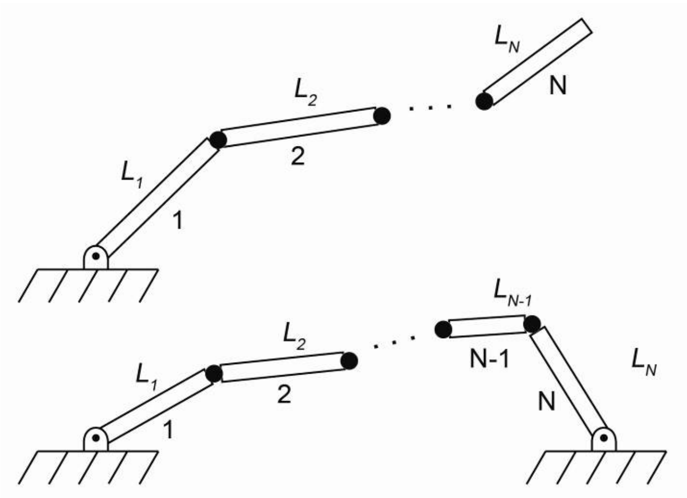


Representations

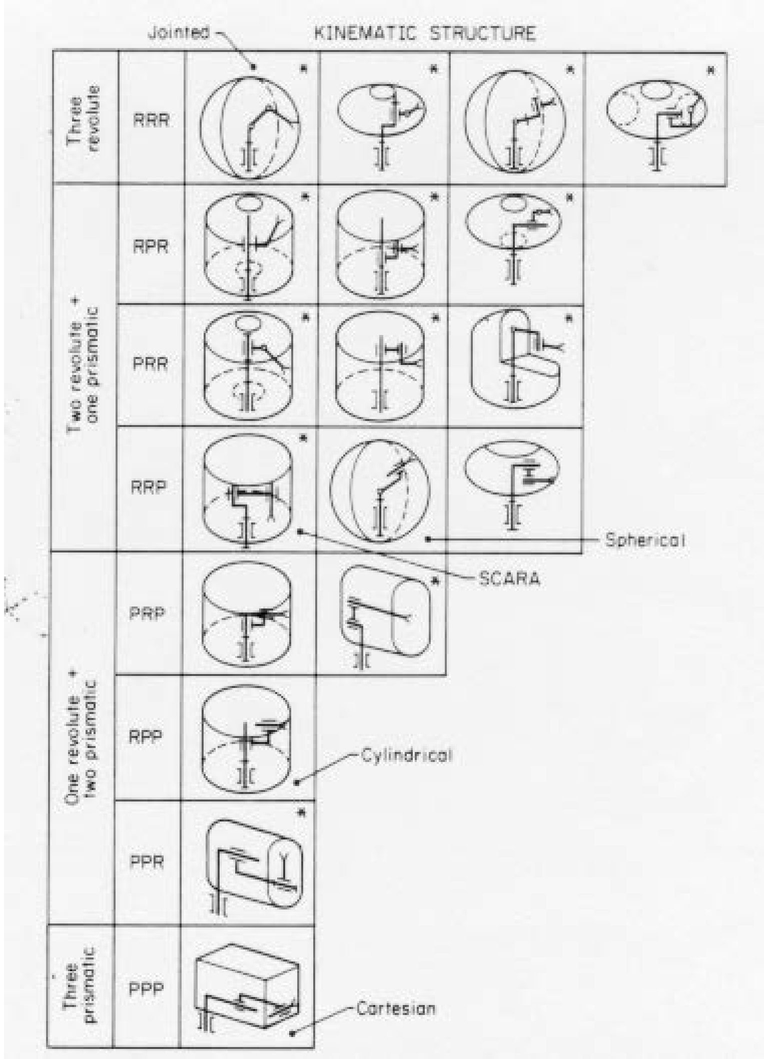
For the majority of this class, we will consider robotic manipulators as open or closed chains of links and joints

Two types of joints: revolute (θ) and prismatic (*d*)





Arm configurations



The most frequent arm configurations are :

- Open kinematic chains :
  - Jointed articulated or anthropomorphic (human-like arms) (RRR)
  - Spherical (RRP)
  - Scara (RRP)
  - Cylindrical (RPP)
  - Cartesian (PPP)
  - Multi-joined (RRRRRR.....) , Redundant configurations
- Closed kinematic chains

Definitions

End-effector/Tool

Device that is in direct contact with the environment. Usually very task-specific

Configuration

Complete specification of every point on a manipulator

set of all possible configurations is the *configuration space*

For rigid links, it is sufficient to specify the configuration space by the joint angles, $q = [q_1 \quad q_2 \quad \dots \quad q_n]^T$

State space

Current configuration (joint positions *q*) and velocities $\dot{q}$

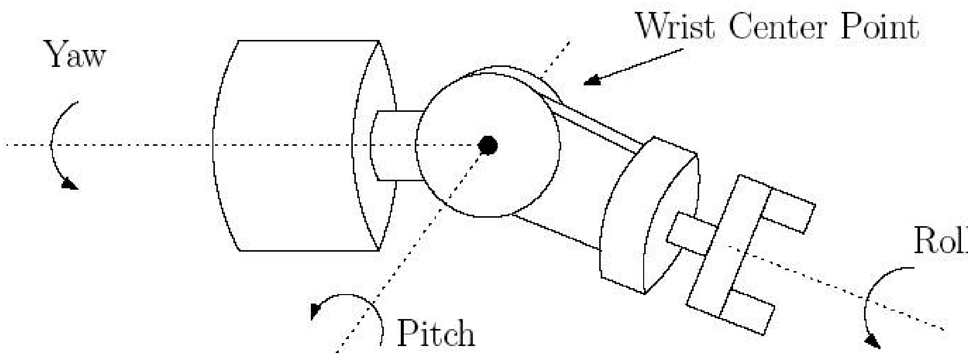
Work space

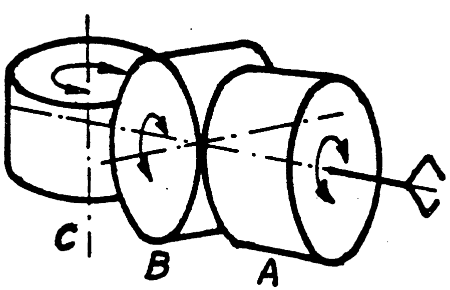
The reachable space the tool can achieve

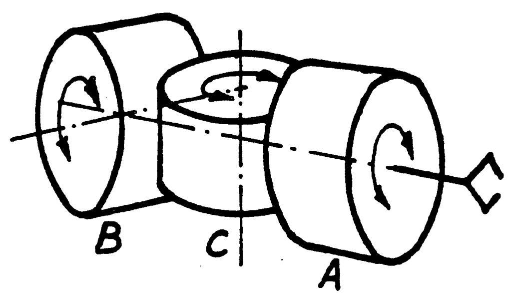
- Reachable workspace
- Dextrous workspace

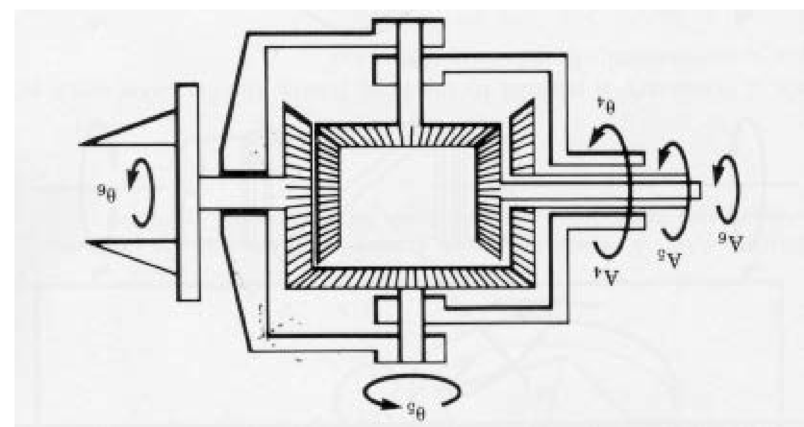
Common configurations: wrists

Many manipulators will be a sequential chain of links and joints forming the ‘arm’ with multiple DOFs concentrated at the ‘wrist’





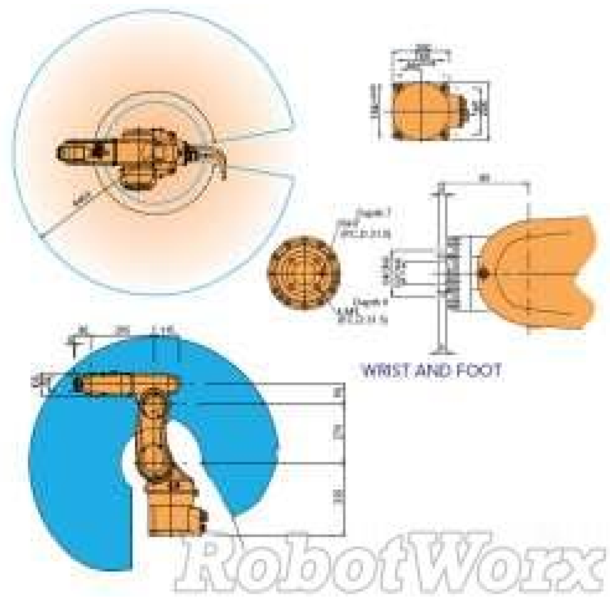




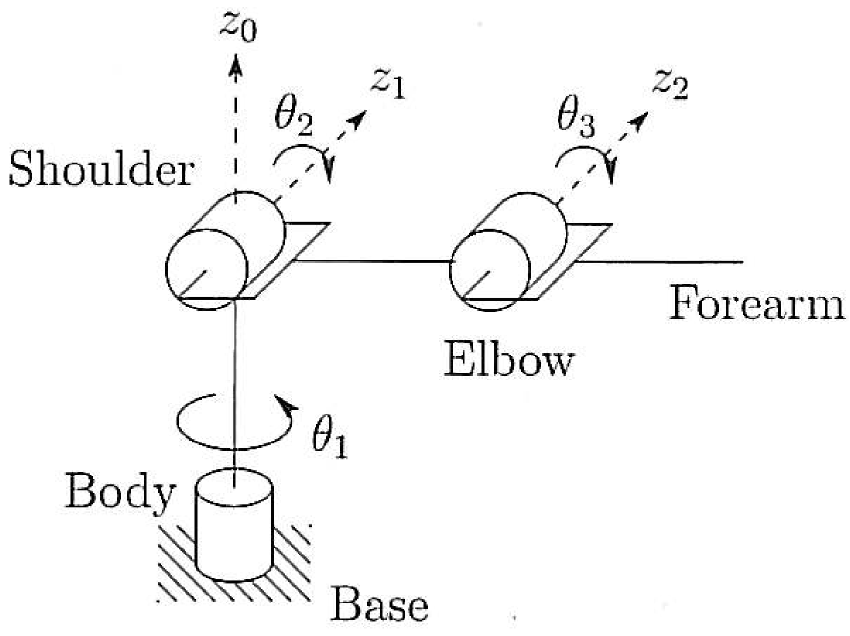
Common configurations: elbow manipulator

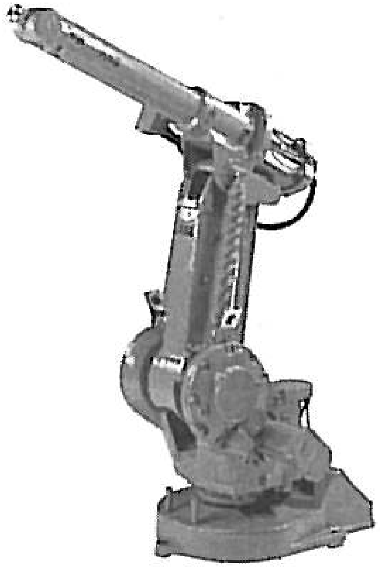
Anthropomorphic arm: ABB IRB1400 or KUKA

Very similar to the lab arm NACHI (RRR)

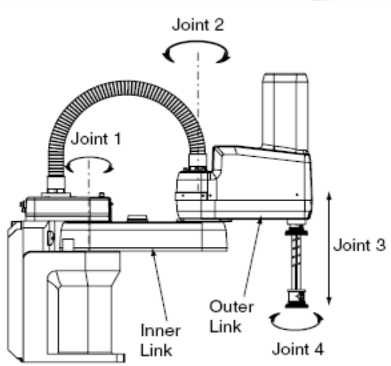


Antropomorphic arm (RRR)





Common configurations: SCARA (RRP)



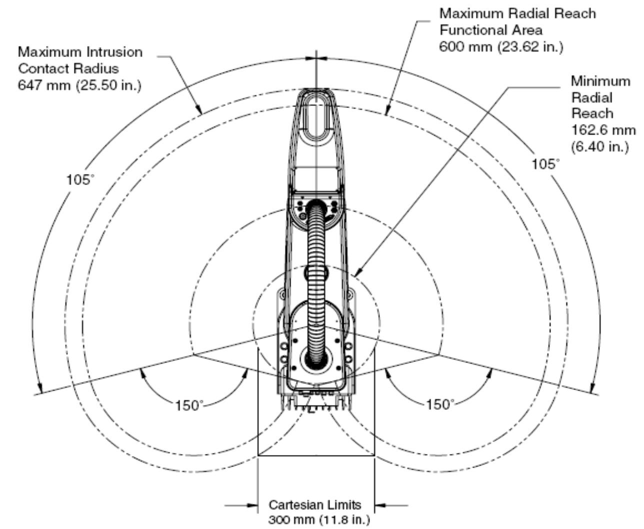


Figure 8-7. Adept Cobra s600 Robot Working Envelope

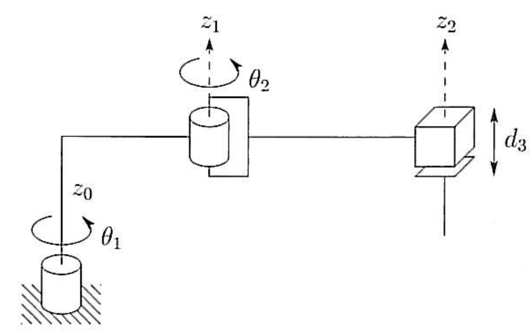
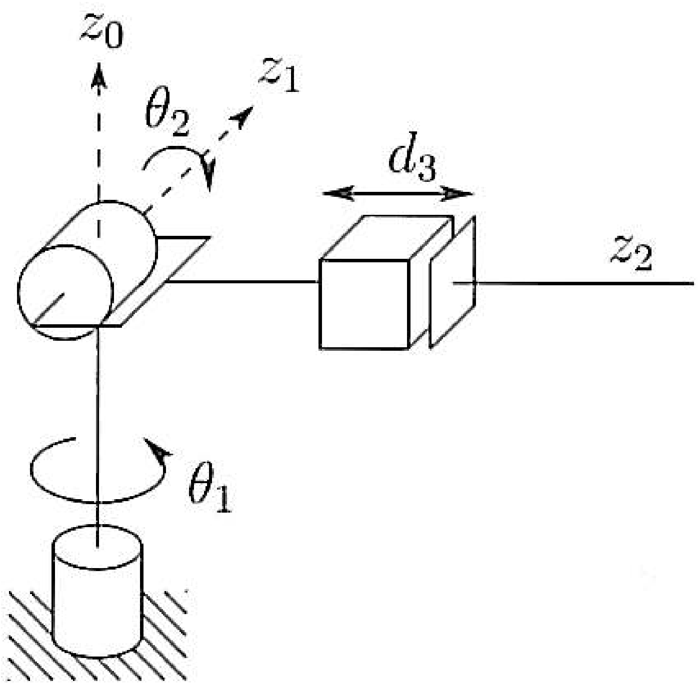
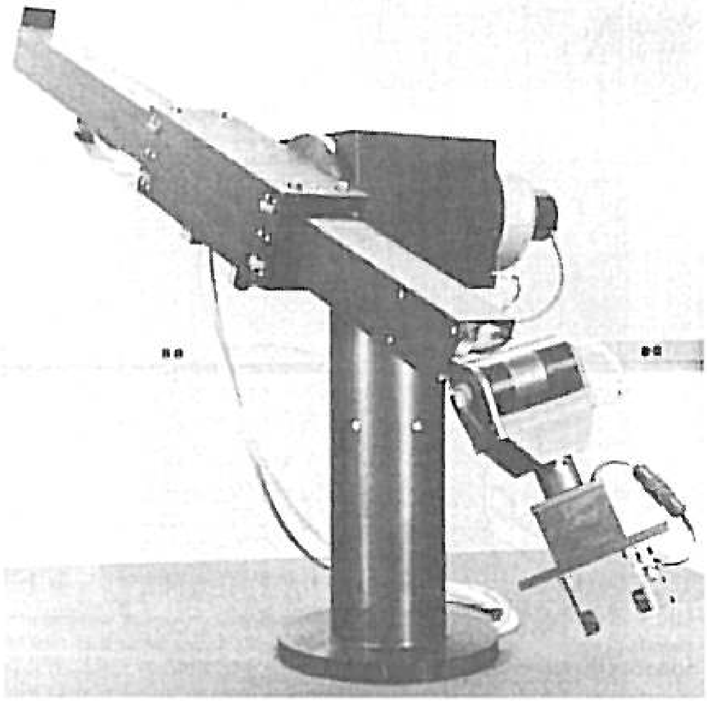


Figure 1.13: Symbolic representation of the SCARA arm.

Spherical Manipulator (RRP)





Common configurations: cylindrical robot (RPP)

workspace forms a cylinder

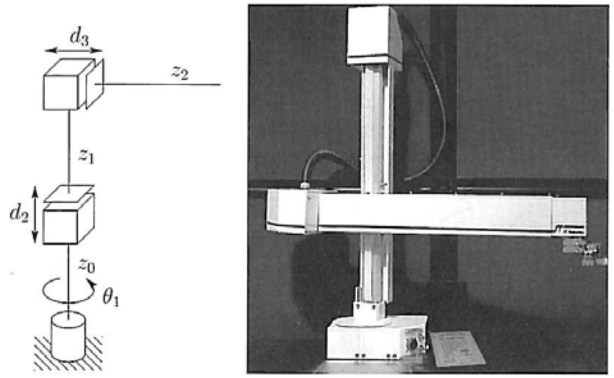


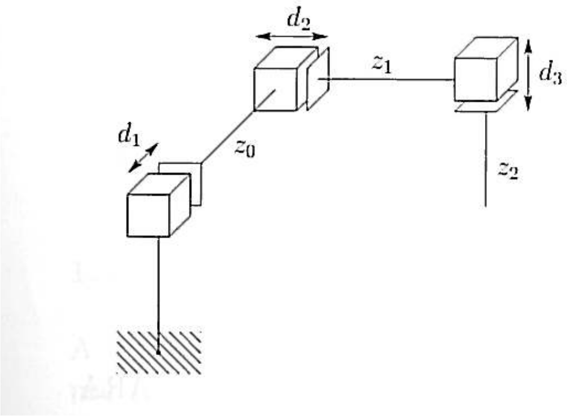
Figure 1.15: The Seiko RT3300 Robot cylindrical robot. Cylindrical robots are often used in materials transfer tasks. (Photo courtesy of Epson Robots.)

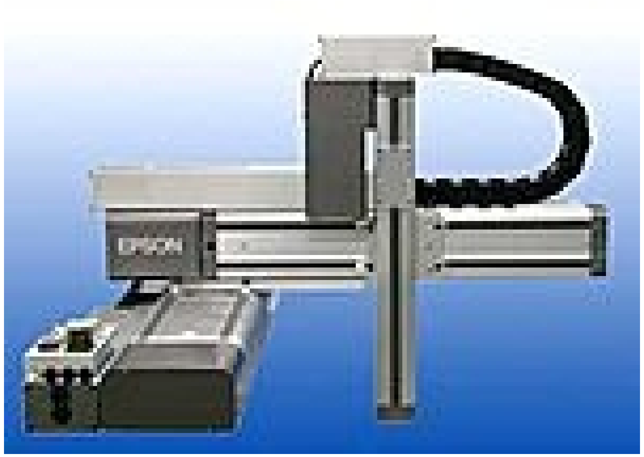
Seiko RT3300 Robot

Common configurations: Cartesian robot (PPP)

Increased structural rigidity, higher precision

Pick and place operations





Epson Chartesian robot (EZ-modules)

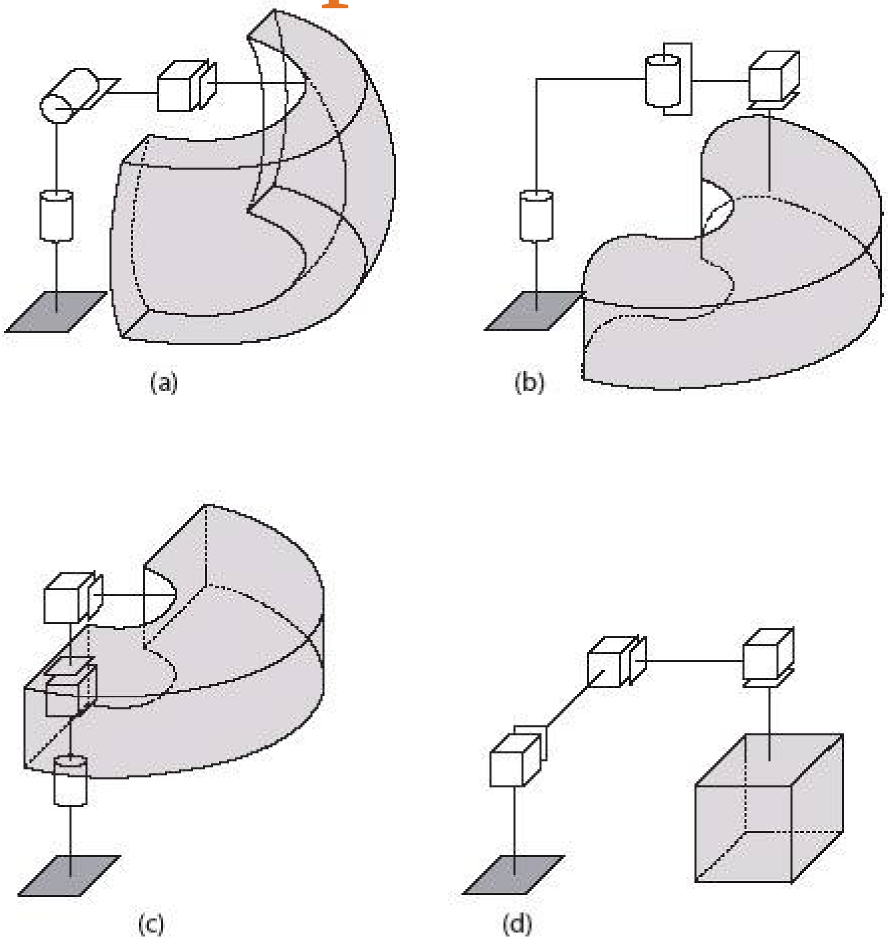
Workspace comparison

(a) spherical

(b) SCARA

(c) cylindrical

(d) Cartesian



Parallel manipulators

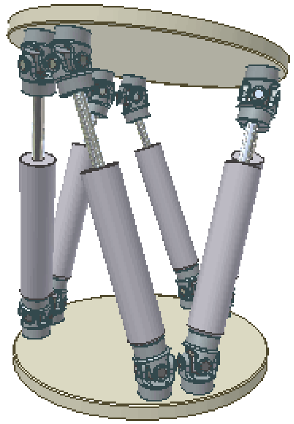
some of the links will form a closed chain with ground

Advantages:

Motors can be proximal: less powerful, higher bandwidth, easier to control

Disadvantages:

Generally less motion, kinematics can be challenging

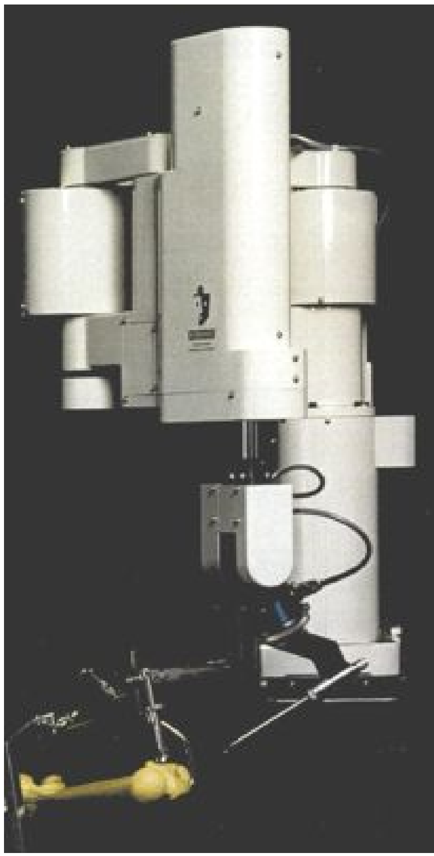


6DOF Stewart platform

ABB IRB940 Tricept

Image-guided robots

ROBODOC –
Integrated Surgical Systems Inc.

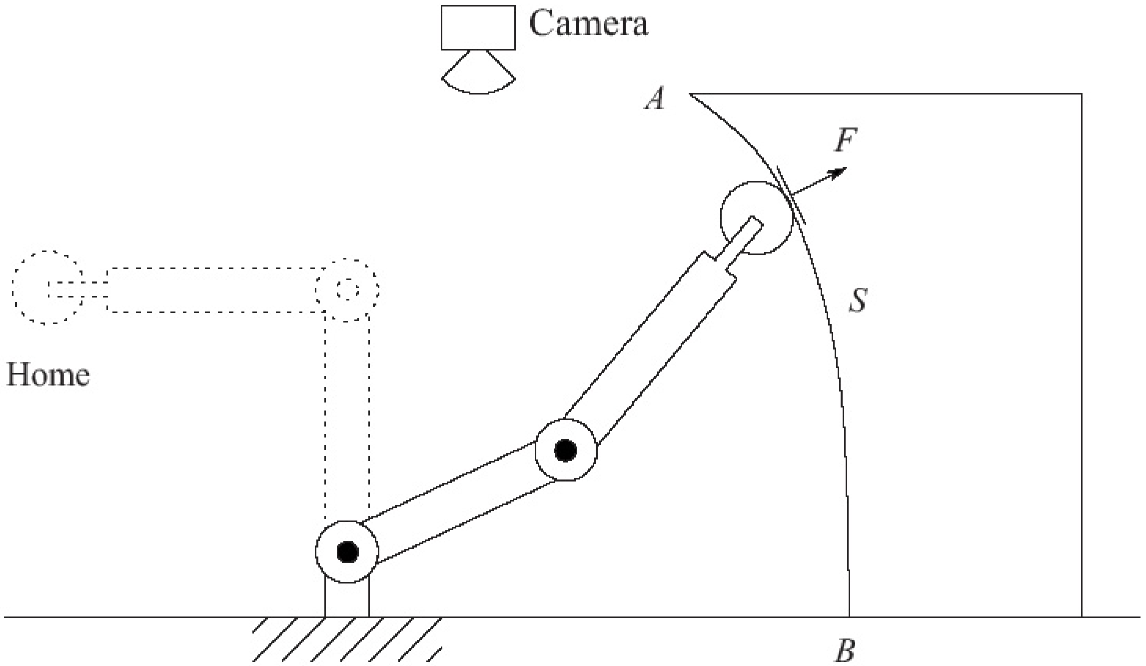


PathFinder –
Armstrong HealthCare Lmt.

CASPAR - Maquet

Simple example: control of a 2DOF planar manipulator

Move from 'home' position and follow the path AB with a constant contact force *F* all using visual feedback



Coordinate frames & forward kinematics

Three coordinate frames: ⓪ ① ②

Positions:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a_1\cos(\theta_1) \\ a_1\sin(\theta_1) \end{bmatrix}$$

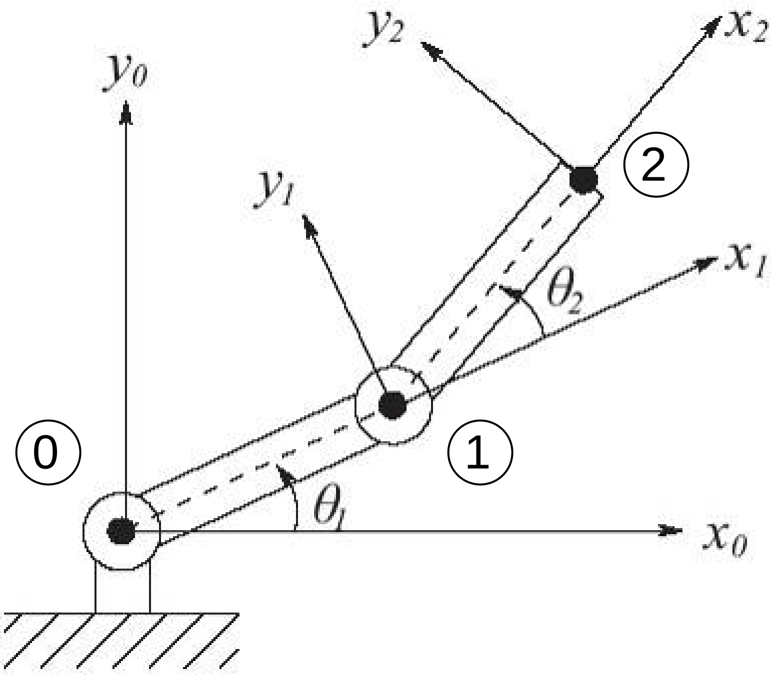
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1\cos(\theta_1) + a_2\cos(\theta_1+\theta_2) \\ a_1\sin(\theta_1) + a_2\sin(\theta_1+\theta_2) \end{bmatrix} \equiv \begin{bmatrix} x \\ y \end{bmatrix}_t$$

$$\hat{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Orientation of the tool frame:

$$\hat{x}_2 = \begin{bmatrix} \cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) \end{bmatrix}, \hat{y}_2 = \begin{bmatrix} -\sin(\theta_1+\theta_2) \\ \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$R_2^0 = \begin{bmatrix} \hat{x}_2\cdot\hat{x}_0 & \hat{y}_2\cdot\hat{x}_0 \\ \hat{x}_2\cdot\hat{y}_0 & \hat{y}_2\cdot\hat{y}_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$



Ch. 2: Rigid Body Motions and Homogeneous Transforms

Representing position

Definition: coordinate frame

A set n of orthonormal basis vectors spanning $\mathbf{R}^n$

For example, $\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

When representing a point p, we need to specify a coordinate frame

With respect to o_0: $p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

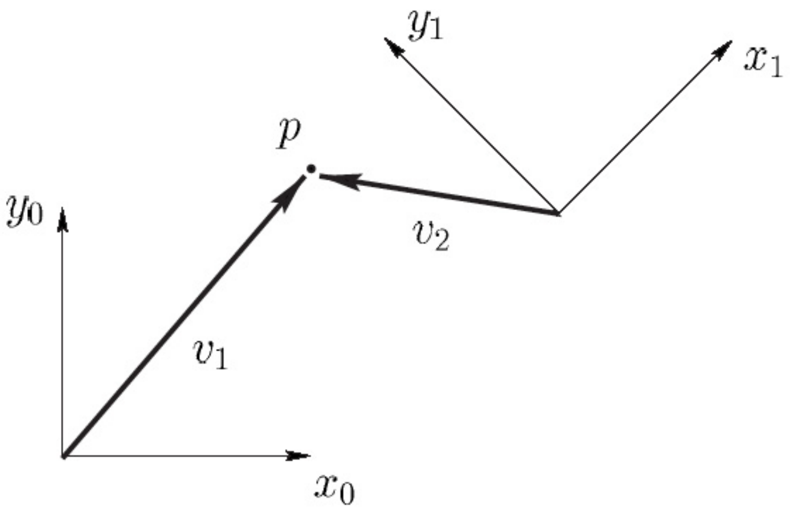
With respect to o_1: $p^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$

v_1 and v_2 are invariant geometric entities

But the representation is dependant

upon choice of coordinate frame

$$v_1^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, v_1^1 = \begin{bmatrix} 7.77 \\ 0.8 \end{bmatrix}, v_2^0 = \begin{bmatrix} -5.1 \\ 1 \end{bmatrix}, v_2^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$



Rotations

2D rotations

Representing one coordinate frame in terms of another

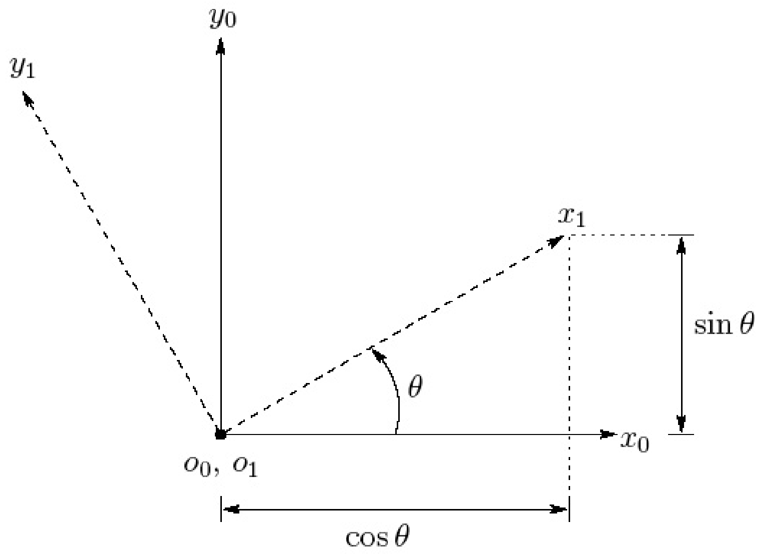
$$R_1^0 = \begin{bmatrix} x_1^0 & y_1^0 \end{bmatrix}$$

Where the unit vectors are defined as:

$$x_1^0 = \|\hat{x}_0\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, y_1^0 = \|\hat{y}_0\| \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This is a **rotation matrix**



Alternate approach

Rotation matrices as projections

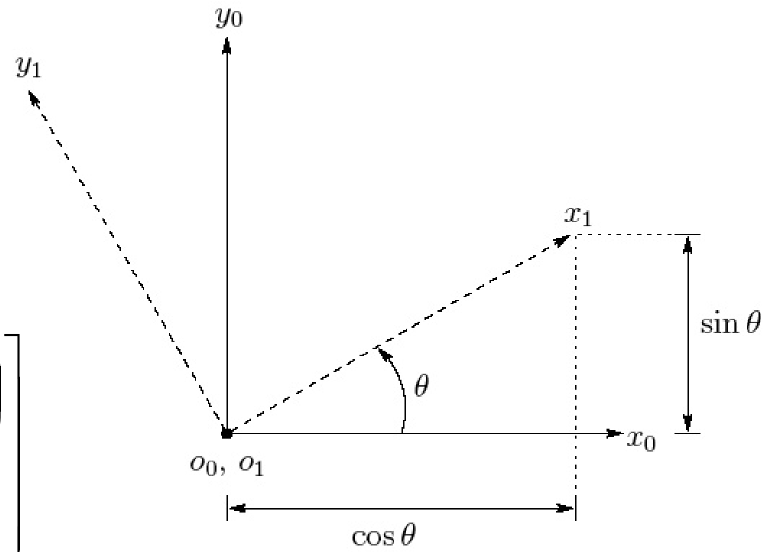
Projecting the axes of from o_1 onto the axes of frame o_0

$$x_1^0 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 \end{bmatrix}, y_1^0 = \begin{bmatrix} \hat{y}_1 \cdot \hat{x}_0 \\ \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix}$$

$$= \begin{bmatrix} \|\hat{x}_1\|\|\hat{x}_0\|\cos\theta & \|\hat{y}_1\|\|\hat{x}_0\|\cos\left(\theta + \frac{\pi}{2}\right) \\ \|\hat{x}_1\|\|\hat{y}_0\|\cos\left(\frac{\pi}{2} - \theta\right) & \|\hat{y}_1\|\|\hat{y}_0\|\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Properties of rotation matrices

Inverse rotations:

$$R_0^1 = \begin{bmatrix} \hat{x}_0 \cdot \hat{x}_1 & \hat{y}_0 \cdot \hat{x}_1 \\ \hat{x}_0 \cdot \hat{y}_1 & \hat{y}_0 \cdot \hat{y}_1 \end{bmatrix}$$

$$= \begin{bmatrix} \|\hat{x}_0\|\|\hat{x}_1\|\cos\theta & \|\hat{y}_0\|\|\hat{x}_1\|\cos\left(\frac{\pi}{2} - \theta\right) \\ \|\hat{x}_0\|\|\hat{y}_1\|\cos\left(\theta + \frac{\pi}{2}\right) & \|\hat{y}_0\|\|\hat{y}_1\|\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \left(R_1^0\right)^T$$

Or, another interpretation uses odd/even properties:

$$R_0^1 = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \left(R_1^0\right)^T$$

Properties of rotation matrices

Inverse of a rotation matrix:

$$\left(R_1^0\right)^{-1} = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \|\hat{x}_1\|\|\hat{x}_0\|\cos\theta & \|\hat{y}_1\|\|\hat{x}_0\|\cos\left(\theta - \frac{\pi}{2}\right) \\ \|\hat{x}_1\|\|\hat{y}_0\|\cos\left(\frac{\pi}{2} - \theta\right) & \|\hat{y}_1\|\|\hat{y}_0\|\cos\theta \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\det\left(R_1^0\right)} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \left(R_1^0\right)^T$$

The determinant of a rotation matrix is always ±1

+1 if we only use right-handed convention

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Properties of rotation matrices

Summary:

Columns (rows) of R are mutually orthogonal

Each column (row) of R is a unit vector

$$R^T = R^{-1}$$
$$\det(R) = 1$$

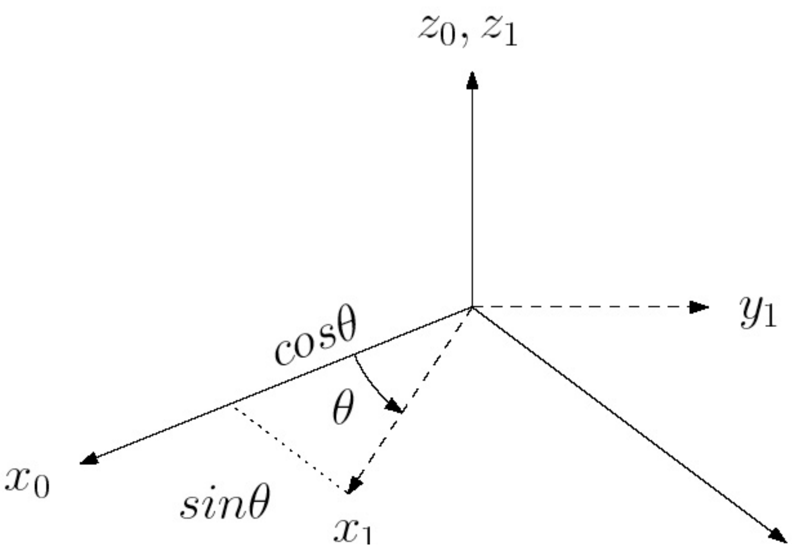
The set of all *n* x *n* matrices that have these properties are called the **Special Orthogonal group** of order *n*

$$R \in SO(n)$$

3D rotations

General 3D rotation:

$$R_1^0 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix} \in SO(3)$$



Special cases

Basic rotation matrices

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of rotation matrices (cont'd)

$SO(3)$ is a group under multiplication

Closure: $\textbf{if } R_1, R_2 \in SO(3) \Rightarrow R_1 R_2 \in SO(3)$

Identity:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO(3)$$

Inverse: $R^T = R^{-1}$

Associativity: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$ → Allows us to combine rotations:

$$R_{ac} = R_{ab} R_{bc}$$

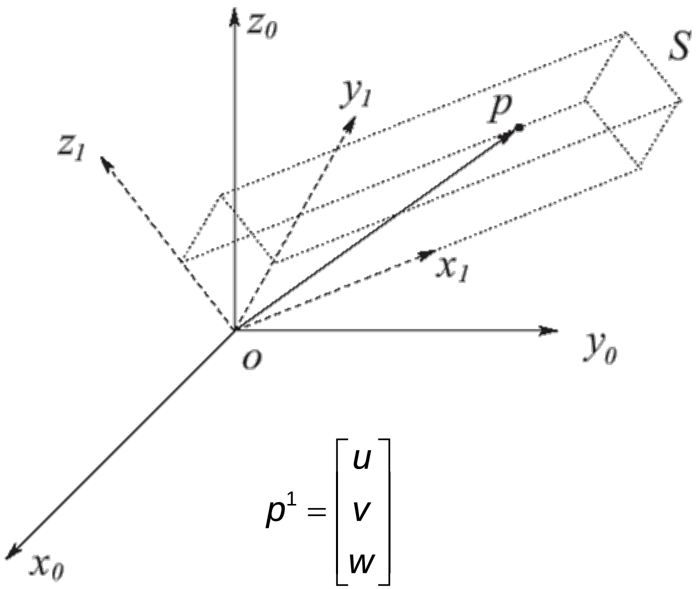
In general, members of $SO(3)$ do not commute

$$R_1 R_2 \neq R_2 R_1$$

Rotational transformations

Now assume p is a fixed point on the rigid object with fixed coordinate frame o_1

The point p can be represented in the frame o_0 (p^0) again by the projection onto the base frame



$$p^1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$p^0 = \begin{bmatrix} p^1 \cdot \hat{x}_0 \\ p^1 \cdot \hat{y}_0 \\ p^1 \cdot \hat{z}_0 \end{bmatrix}$$

$$= \begin{bmatrix} (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{x}_0 \\ (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{y}_0 \\ (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{z}_0 \end{bmatrix}$$

$$= \begin{bmatrix} u\hat{x}_1 \cdot \hat{x}_0 + v\hat{y}_1 \cdot \hat{x}_0 + w\hat{z}_1 \cdot \hat{x}_0 \\ u\hat{x}_1 \cdot \hat{y}_0 + v\hat{y}_1 \cdot \hat{y}_0 + w\hat{z}_1 \cdot \hat{y}_0 \\ u\hat{x}_1 \cdot \hat{z}_0 + v\hat{y}_1 \cdot \hat{z}_0 + w\hat{z}_1 \cdot \hat{z}_0 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_1^0 p^1$$

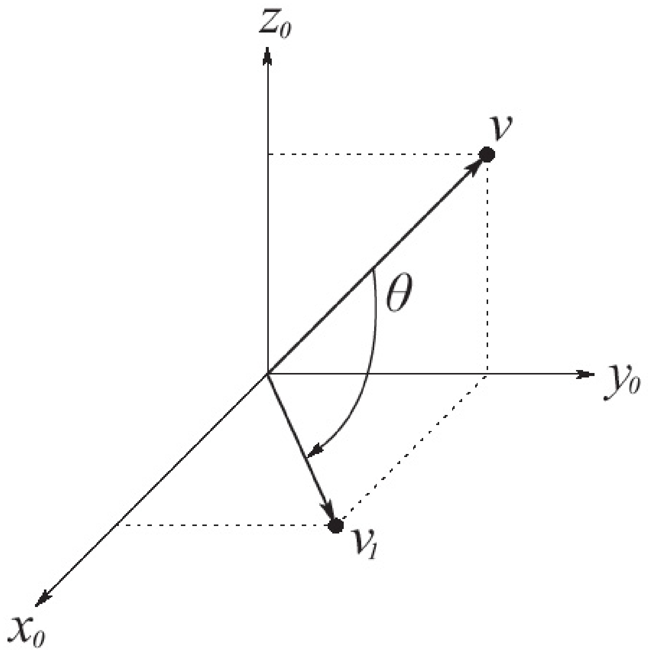
Rotating a vector

Another interpretation of a rotation matrix:

Rotating a vector about an axis in a fixed frame

Ex: rotate v^0 about y_0 by $\pi/2$

$$v^0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$



$$\begin{aligned} v^1 &= R_{y,\pi/2} v^0 \\ &= \left.\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\right|_{\theta=\pi/2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Rotation matrix summary

Three interpretations for the role of rotation matrix:

- Representing the coordinates of a point in two different frames
- Orientation of a transformed coordinate frame with respect to a fixed frame
- Rotating vectors in the same coordinate frame

Similarity transforms

All coordinate frames are defined by a set of basis vectors

- These span $\boldsymbol{R}^n$
- Ex: the unit vectors i, j, k

In linear algebra, a n x n matrix A is a mapping from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$

- $y = Ax$, where y is the image of x under the transformation A
- Think of x as a linear combination of unit vectors (basis vectors), for example the unit vectors:

$$e_1 = [1 \quad 0 \quad \ldots \quad 0]^T, \ldots, e_n = [0 \quad 0 \quad \ldots \quad 1]^T$$

- Then the columns of A are the images of these basis vectors
- If we want to represent vectors with respect to a different basis, e.g.: $f_1, \ldots, f_n$, the transformation A can be represented by:

$$A' = T^{-1} A T$$

- Where the columns of T are the vectors $f_1, \ldots, f_n$,

Similarity transforms

A' and *A* have the same eigenvalues

An eigenvector *x* of *A* corresponds to an eigenvector $T^{-1}x$ of *A'*

Rotation matrices are also a change of basis

If *A* is a linear transformation in o_0 and *B* is a linear transformation in o_1, then they are related as follows:

$$B = \left(R_1^0\right)^{-1} A R_1^0$$

Ex: the frames o_0 and o_1 are related as follows:

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

If the matrix *A* is also a rotation matrix $R_{z,\theta}$ (relative to o_0) the rotation expressed in o_1 is:

$$B = \left(R_1^0\right)^{-1} A R_1^0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Compositions of rotations

w/ respect to the current frame

Ex: three frames o_0, o_1, o_2

$$\left.\begin{aligned} p^0 &= R_1^0 p^1 \\ p^1 &= R_2^1 p^2 \\ p^0 &= R_2^0 p^2 \end{aligned}\right\} p^0 = R_1^0 R_2^1 p^2 \longrightarrow R_2^0 = R_1^0 R_2^1$$

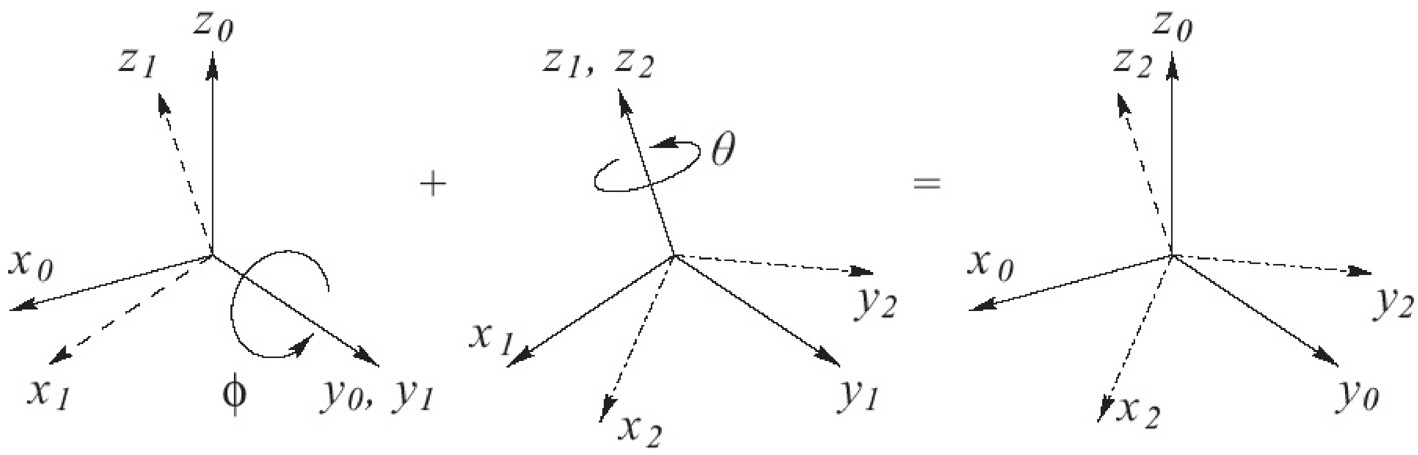
This defines the composition law for successive rotations about the **current** reference frame: post-multiplication

Compositions of rotations

Ex: R represents rotation about the current y-axis by ϕ followed by θ about the current z-axis

$$R = R_{y,\phi} R_{z,\theta}$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & -\cos\phi\sin\theta & \sin\phi \\ \sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{bmatrix}$$



Compositions of rotations

w/ respect to a fixed reference frame (o_0)

Let the rotation between two frames o_0 and o_1 be defined by R_1^0

Let R be a desired rotation w/ respect to the fixed frame o_0

Using the definition of a similarity transform, we have:

$$R_2^0 = R_1^0\left[\left(R_1^0\right)^{-1} R R_1^0\right] = R R_1^0$$

This defines the composition law for successive rotations about a **fixed** reference frame: pre-multiplication

Compositions of rotations

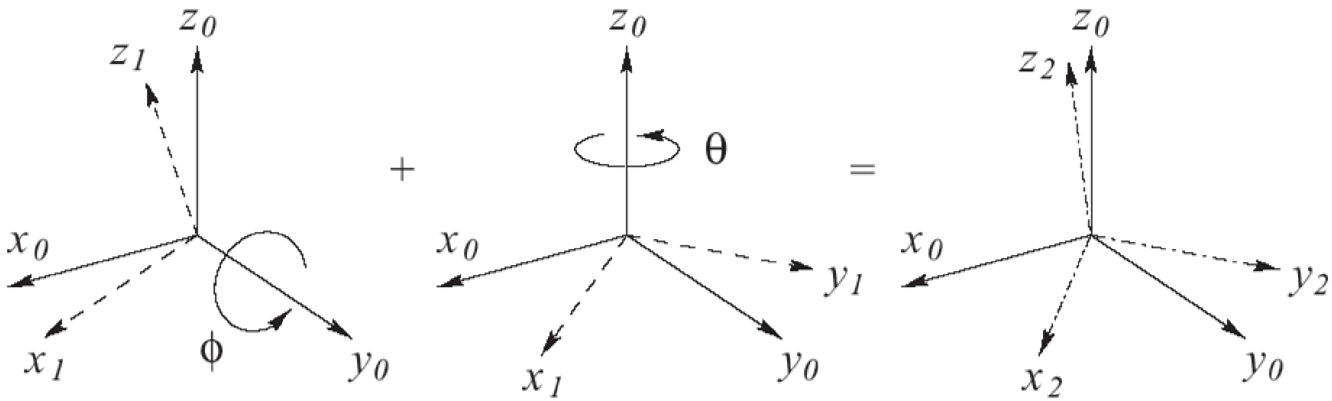
Ex: we want a rotation matrix R that is a composition of ϕ about y_0 ($R_{y,\phi}$) and then θ about z_0 ($R_{z,\theta}$)

the second rotation needs to be projected back to the initial fixed frame

$$R_2^0 = \left(R_{y,\theta}\right)^{-1} R_{z,\theta} R_{y,\theta}$$
$$= R_{y,-\theta} R_{z,\theta} R_{y,\theta}$$

Now the combination of the two rotations is:

$$R = R_{y,\phi}\left[R_{y,-\phi} R_{z,\theta} R_{y,\phi}\right] = R_{z,\theta} R_{y,\phi}$$



Compositions of rotations

Summary:

- Consecutive rotations w/ respect to the current reference frame:
  - Post-multiplying by successive rotation matrices
- w/ respect to a fixed reference frame (o_0)
  - Pre-multiplying by successive rotation matrices
- We can also have hybrid compositions of rotations with respect to the current and a fixed frame using these same rules

Parameterizing rotations

There are three parameters that need to be specified to create arbitrary rigid body rotations

We will describe three such parameterizations:

Euler angles

Roll, Pitch, Yaw angles

Axis/Angle

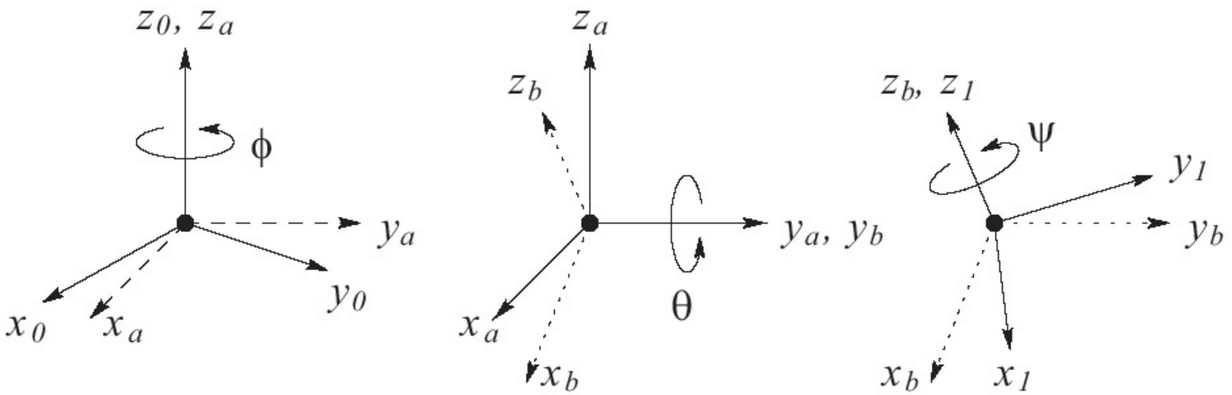
Parameterizing rotations

Euler angles

Rotation by ϕ about the z-axis, followed by θ about the current y-axis, then ψ about the current z-axis

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{z,\psi} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

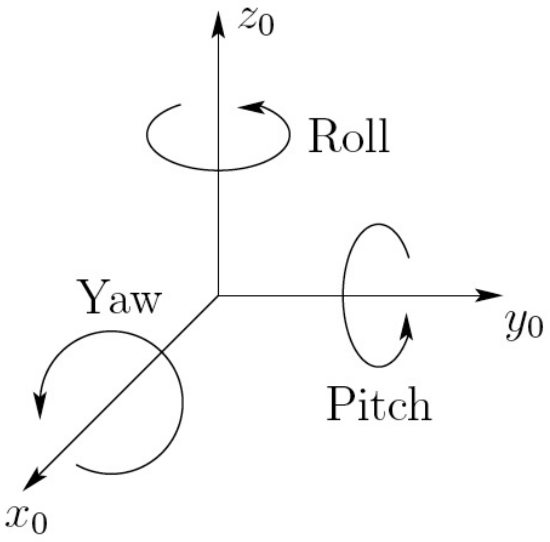


Parameterizing rotations

Roll, Pitch, Yaw angles

Three consecutive rotations about the fixed principal axes:

Yaw (x_0) ψ, pitch (y_0) θ, roll (z_0) ϕ



$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

Parameterizing rotations

Axis/Angle representation

Any rotation matrix in SO(3) can be represented as a single rotation about a suitable axis through a set angle

For example, assume that we have a unit vector:

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$$

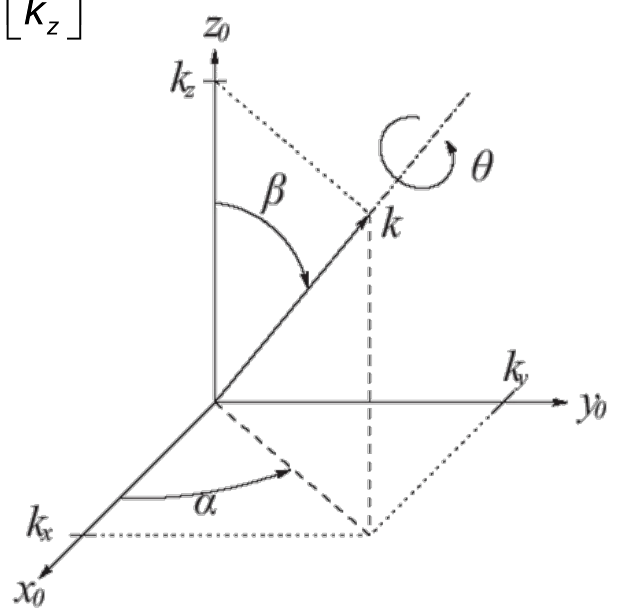
Given θ, we want to derive $R_{k,\theta}$:

Intermediate step: project the *z*-axis onto *k*:

$$R_{k,\theta} = R R_{z,\theta} R^{-1}$$

Where the rotation *R* is given by:

$$R = R_{z,\alpha} R_{y,\beta}$$
$$\Rightarrow R_{k,\theta} = R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}$$



Parameterizing rotations

Axis/Angle representation

This is given by:

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

Inverse problem:

Given arbitrary *R*, find *k* and θ

$$\theta = \cos^{-1}\left(\frac{Tr(R)-1}{2}\right)$$

$$\hat{k} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Alternative approach

Rotation matrices in SO(3) can be derived using an alternate method using the matrix exponential

Assume that you have a unit vector, ω, that represents an axis of rotation

Take a vector *q*(*t*) and rotate it about ω with unity velocity

This gives:

$$\dot{\vec{q}}(t) = \omega \times \vec{q}(t) = \hat{\omega}\vec{q}(t)$$

Where the 'cross-product' matrix is given as follows:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Now integrate to find the vector *q*(*t*):

Rotating for θ units of time gives:

$$\vec{q}(t) = e^{\hat{\omega}t}\vec{q}(0)$$

$$\vec{q}(t) = e^{\hat{\omega}\theta}\vec{q}(0) = R_{\omega,\theta}\vec{q}(0)$$

Thus the rotation about ω can be represented by the matrix exponential

For notation, we can call:

$$\hat{\omega} \in so(3)$$

Rigid motions

Rigid motion is a combination of rotation and translation

Defined by a rotation matrix (*R*) and a displacement vector (*d*)

$$R \in SO(3)$$
$$d \in \mathbf{R}^3$$

the group of all rigid motions (*d*,*R*) is known as the **Special Euclidean group**, *SE*(3)

$$SE(3) = \mathbf{R}^3 \times SO(3)$$

Consider three frames, o_0, o_1, and o_2 and corresponding rotation matrices R_2^1, and R_1^0

Let d_2^1 be the vector from the origin o_1 to o_2, d_1^0 from o_0 to o_1

For a point p^2 attached to o_2, we can represent this vector in frames o_0 and o_1:

$$\begin{aligned} p^1 &= R_2^1 p^2 + d_2^1 \\ p^0 &= R_1^0 p^1 + d_1^0 \\ &= R_1^0 \left(R_2^1 p^2 + d_2^1 \right) + d_1^0 \\ &= R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0 \end{aligned}$$

Homogeneous transforms

We can represent rigid motions (rotations and translations) as matrix multiplication

Define:

$$H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix}$$

Now the point p_2 can be represented in frame o_0:

$$P^0 = H_1^0 H_2^1 P^2$$

Where the P^0 and P^2 are:

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, P^2 = \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$$

Homogeneous transforms

The matrix multiplication *H* is known as a **homogeneous transform** and we note that

$$H \in SE(3)$$

Inverse transforms:

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

Homogeneous transforms

Basic transforms:

Three pure translation, three pure rotation

$$\mathbf{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$