# Undecidability and Complexity in Five Lectures

## Overview

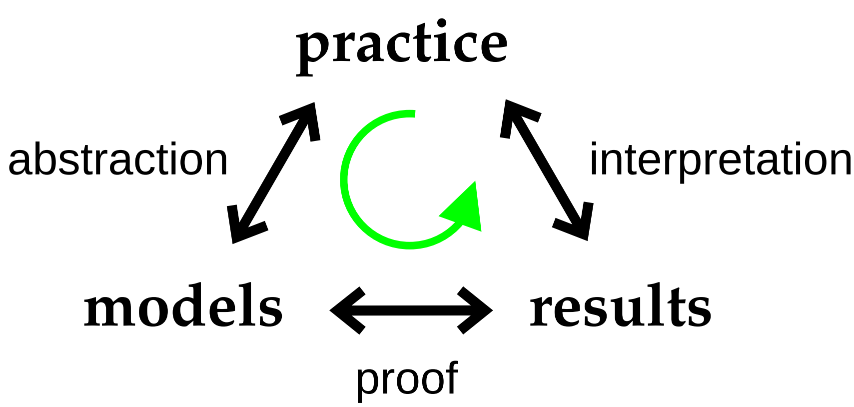
- Lecture 1: Modeling Problems and Solutions.
- Lecture 2: Unsolvability
- Lecture 3: Intractability.
- Lecture 4: Proving Intractability.
- Lecture 5: Coping With Intractability.

## Lecture 1 overview

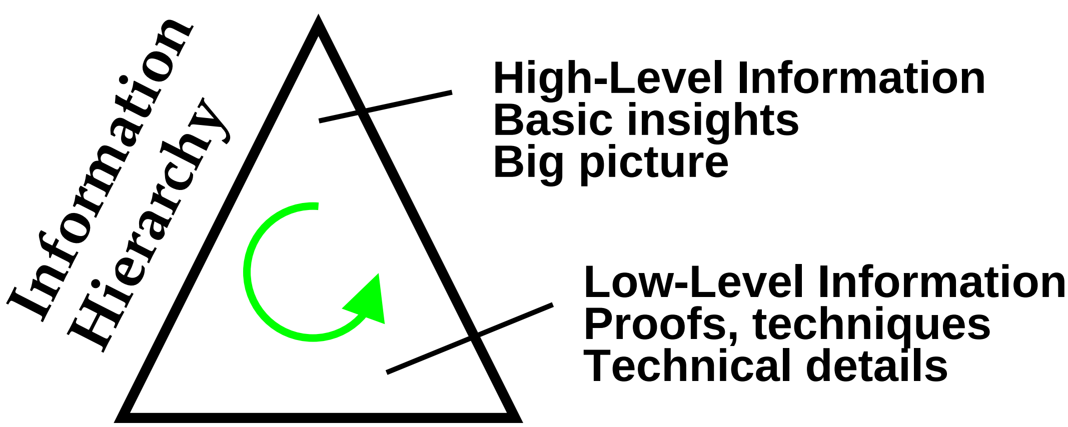
- Our approach - modeling
- The subject matter - what is this all about
- Historical introduction
- How to model problems
- How to model solutions

# Our approach

## Modeling




## Perspective




Lectures → Mainly high-level understanding

Group sessions → Practice skills: proofs, problems

Studying strategy: Don't memorize pensum – try to understand the whole!

# Subject matter

How to **solve** information-processing **problems efficiently**.

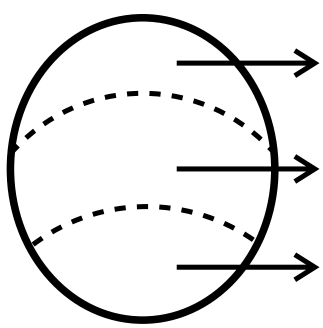
abstraction
formalisation
modeling

| | | | | |
|---|---|---|---|---|
| Problems | $\rightsquigarrow$ | interesting, natural problems | $\rightsquigarrow$ | formal languages (F.L.s) |

(Ex. MATCHING, SORTING, T.S.P.)

| | | | | |
|---|---|---|---|---|
| Solutions | $\rightsquigarrow$ | algorithms | $\rightsquigarrow$ | Turing machines |
| Efficiency | $\rightsquigarrow$ | complexity | $\rightsquigarrow$ | complexity classes |

Problems,
F.L.s



Unsolvable (impossible)
Intractable (horrible)
Nice

# Historical introduction

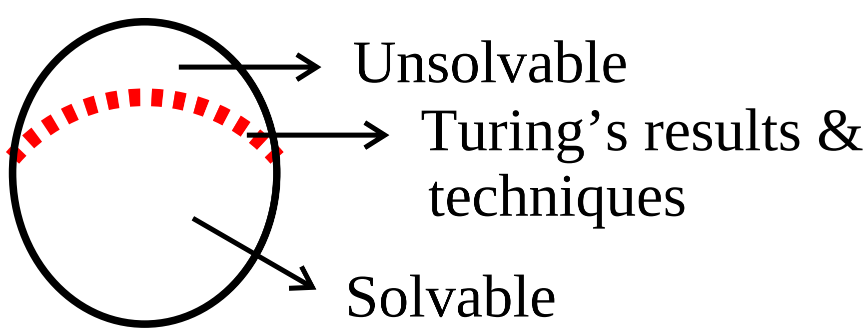
In mathematics (cooking, engineering, life)
solution = algorithm

Examples:

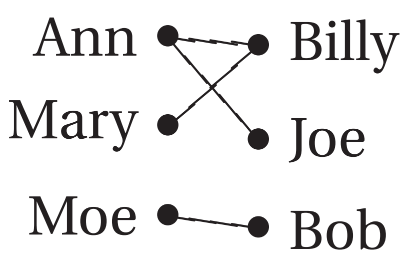
- $\sqrt{253} =$
- $ax^2 + bx + c = 0$
- Euclid's g.c.d. algorithm — the earliest non-trivial algorithm?

$\exists$ algorithm? $\rightarrow$ metamathematics

- K. Gödel (1931): nonexistent theories
- A. Turing (1936): nonexistent algorithms (article: "On computable Numbers . . . ")



- Von Neumann (ca. 1948): first computer
- Edmonds (ca. 1965): an algorithm for MAXIMUM MATCHING




Edmonds' article rejected based on existence of trivial algorithm: Try all possibilities!

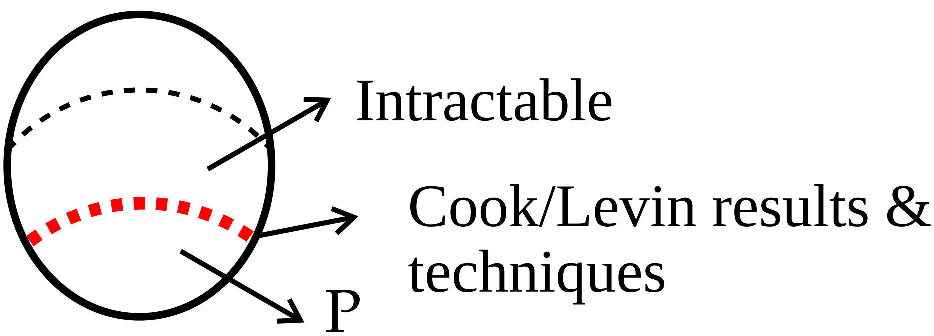
**Complexity analysis of trivial algorithm** (using approximation)

- $n = 100$ boys
- $n! = 100 \times 99 \times \cdots \times 1 \geq 10^{90}$ possibilities
- assume $\leq 10^{12}$ possibilites tested per second
- $\leq 10^{12+4+2+3+2} \leq 10^{23}$ tested per century
- running time of trivial algorithm for $n = 100$ is $\geq 10^{90-23} = 10^{67}$ centuries!

Compare: "only" ca. $10^{13}$ years since Big Bang!

Edmonds: My algorithm is a **polynomial-time** algorithm, the trivial algorithm is **exponential-time**!

- $\exists$ polynomial-time algorithm for a given problem?
- Cook / Levin (1972): $\mathcal{NP}$**-completeness**

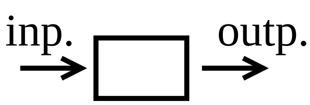


# Problems, formal languages

All the world's information-processing problems — *Ex. compute salaries, control Lunar module landing*

graphs, numbers ...

"Interesting", "natural" problems — MATCHING, TSP, SORTING




Functions — *(sets of I/O pairs)*

output= YES/NO

Formal languages — *(sets of 'YES-strings')*

Problem = set of strings (over an alphabet). Each string is (the encoding of) a YES-instance.

**Def. 1** ***Alphabet*** *= finite set of symbols*

Ex. $\sum = \{0, 1\}$ ; $\Sigma = \{A, \ldots, Z\}$

Coding: binary $\leftrightarrow$ ASCII

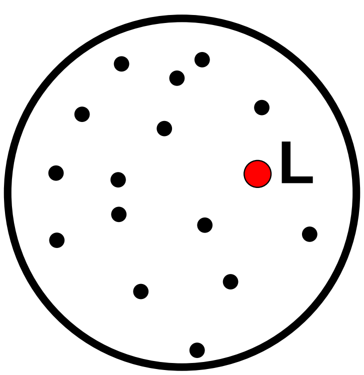
**Def. 2** $\sum^*$ = *all finite strings over* $\sum$

$\sum^* = \{\epsilon, 0, 1, 00, 01, \cdots\}$ — in **lexicographic order**

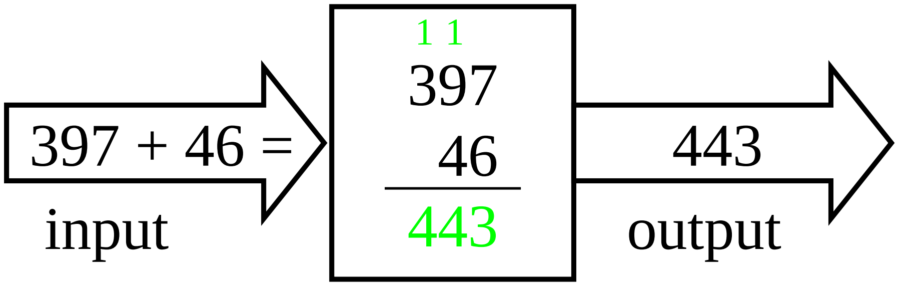
**Def. 3** *A* ***formal language*** $L$ *over* $\sum$ *is a subset of* $\sum^*$

L is the set of all "YES-instances".

Set of all problems

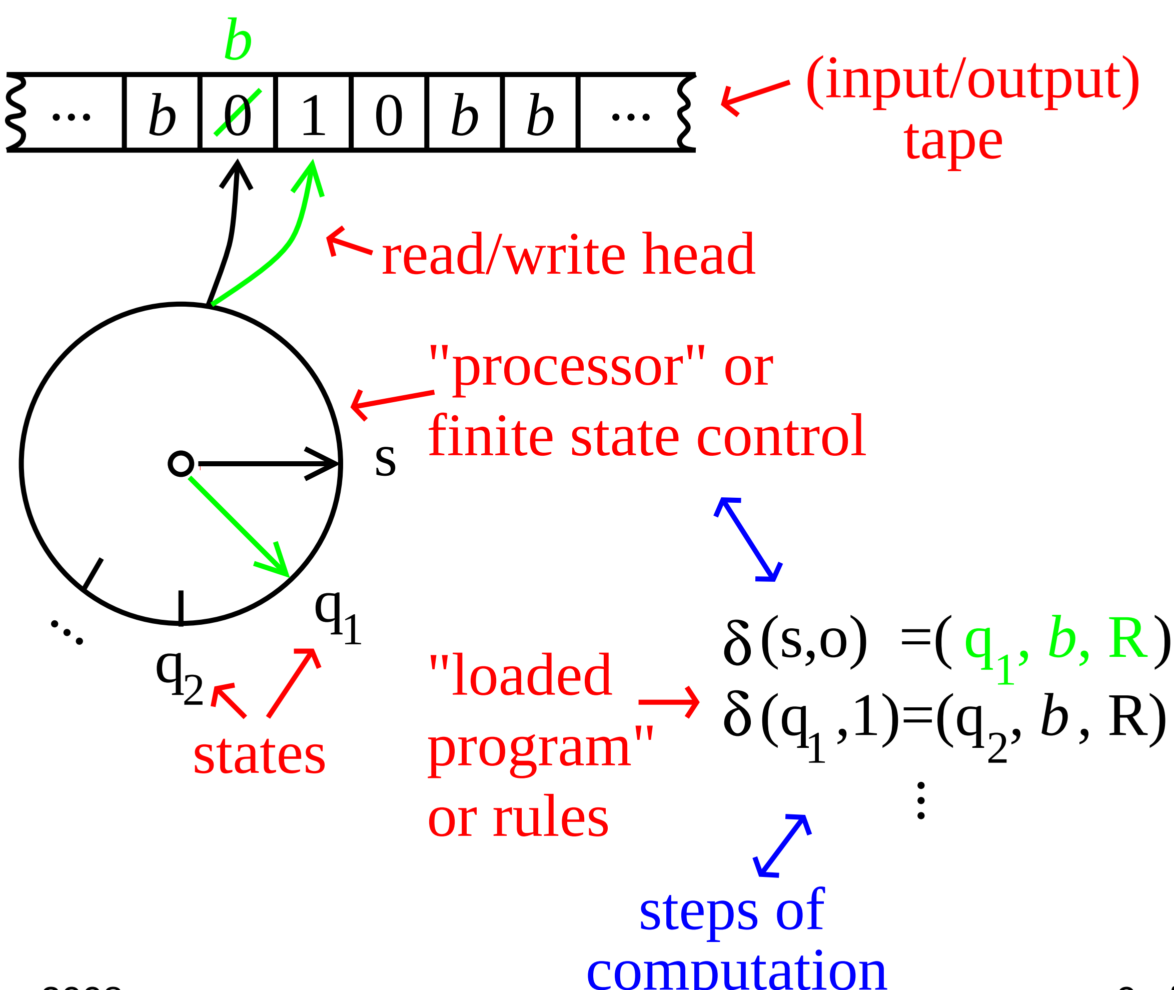


# Algorithm




computation ⟷ rules

# Turing machine – intuitive description



We say that Turing machine $M$ **decides language L** if (and only if) $M$ computes the function

$$f : \Sigma^* \to \{Y, N\} \text{ and } \begin{array}{l} \text{for each } x \in L : f(x) = Y \\ \text{for each } x \notin L : f(x) = N \end{array}$$

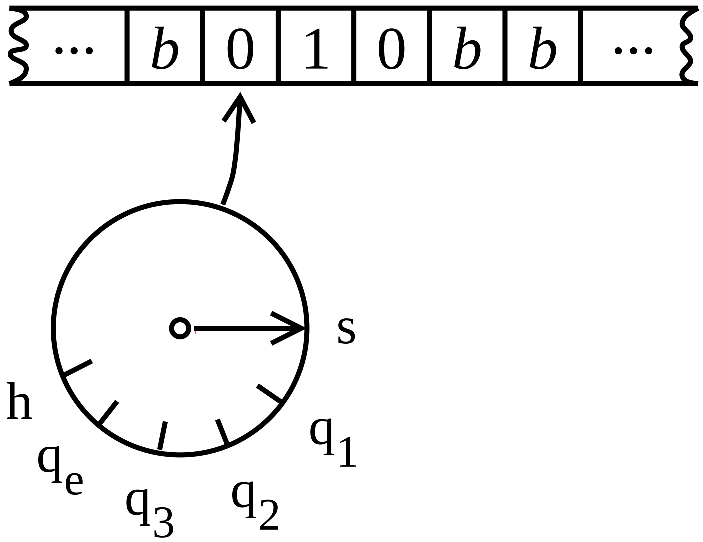
Language L is **(Turing) decidable** if (and only if) there is a Turing machine which decides it.

We say that Turing machine $M$ **accepts language L** if $M$ halts if and only if its input is an string in L.

Language L is **(Turing) acceptable** if (and only if) there is a Turing machine which accepts it.

## Example

A Turing machine $M$ which decides $L = \{010\}$.




$$M = (\Sigma, \Gamma, Q, \delta) \qquad \Sigma = \{0, 1\}$$
$$\Gamma = \{0, 1, b, Y, N\} \qquad Q = \{s, h, q_1, q_2, q_3, q_e\}$$

$\delta$ :

| | $0$ | $1$ | $b$ |
|---|---|---|---|
| $s$ | $(q_1, b, R)$ | $(q_e, b, R)$ | $(h, N, -)$ |
| $q_1$ | $(q_e, b, R)$ | $(q_2, b, R)$ | $(h, N, -)$ |
| $q_2$ | $(q_3, b, R)$ | $(q_e, b, R)$ | $(h, N, -)$ |
| $q_3$ | $(q_e, b, R)$ | $(q_e, b, R)$ | $(h, Y, -)$ |
| $q_e$ | $(q_e, b, R)$ | $(q_e, b, R)$ | $(h, N, -)$ |

('$-$' means "don't move the read/write head")

# Church’s thesis

## ’Turing machine’ $\cong$ ’algorithm’

Turing machines can compute every function that can be computed by some algorithm or program or computer.

## ’Expressive power’ of PL’s

**Turing complete** programming languages.

## ’Universality’ of computer models

Neural networks are Turing complete (Mc Cullok, Pitts).

## Uncomputability

If a Turing machine cannot compute $f$, no computer can!