INF4140, 2012. Exercise 6

Substitutions

Do the following substitutions:

$(x == 2)_{x \leftarrow y}$ $\quad$ $(x == 2)_{x \leftarrow 2}$
$(x == 2)_{x \leftarrow (x+2)}$ $\quad$ $(x == y)_{x \leftarrow (y+1)}$
$(x > y \wedge y \leq 2)_{y \leftarrow (z+2)}$

The assignment axiom

1. The following triples are instances of the assignment axiom:

$$\{0 == 0 \wedge y < z\}\texttt{x} = \texttt{0}\{x = 0 \wedge y < z\}$$
$$\{y - (x+1) < z\}\texttt{x} = \texttt{x} + \texttt{1}\{(y - x) < z\}$$

   Use Definition 2.4 (p. 59) to convince yourself about the trueness of these triples in our computational model.

2. Given the following triples:

$$\{y < z\}\texttt{x} = \texttt{0}\{x = 0 \wedge y < z\}$$
$$\{(y - x) == z\}\texttt{x} = \texttt{x} + \texttt{1}\{y - x < z\}$$

   Convince yourself about the trueness of these triples and use the axiom of assignment to prove them.

3. Consider the following two triples:

$$\{a == 2^k * y\}\texttt{k} = \texttt{k} - \texttt{1}\{a == 2^{k+1} * y\}$$
$$\{a == 2^{k+1} * y\}\texttt{y} = \texttt{y} * 2\{a == 2^k * y\}$$

   Again, use Definition 2.4 from the book to convince yourself about the trueness of these triples. Use the assignment axiom to prove their correctness. Can we use these two triples to say something about the program `k = k -1; y = y * 2`?

4. Def. 2.4 expresses that $\{P\}S\{Q\}$ is true if *all* possible executions of S that starts in a state satisfying P, terminates in a state satisfying Q. Thus, in order to prove that $\{P\}S\{Q\}$ is *not* true, it is enough to find one execution of S starting in a state satisfying P, but terminating in a state *not* satisfying Q. This execution then serves as a *counterexample* for the trueness of $\{P\}S\{Q\}$. Find counterexamples to show that the following two triples are not true:

$$\{x < 5\}\texttt{x} = \texttt{x} + \texttt{1}\{x < 5\}$$
$$\{x \mathrel{!=} y\}\texttt{x} = \texttt{2} * \texttt{x}\{x \mathrel{!=} y\}$$

   Try to apply the axiom of assignment on the two triples. Are we able to prove these triples using the assignment axiom?

Free variables

The book (page 60) defines $P_{x \leftarrow e}$ as P with all *free* occurrences of x replaced by e. A free variable is a variable that is not in the scope of a quantifier $(\forall, \exists)$. For instance, y is *free* in the predicate $\forall x : x \leq y$, but x is not free.

What is the result of the two substitutions $(\forall x : x \leq y)_{x \leftarrow e}$ and $(\exists x : x \leq y)_{y \leftarrow e}$?

Given the predicates:
P: $y > z \wedge (\forall i : 1 \leq i \leq n : a[i] <= max)$
Q: $i > z \wedge (\exists i : 1 \leq i \leq n : a[i] <= max)$

Evaluate $P_{z \leftarrow (z+1)}$, $P_{max \leftarrow (max*2)}$ and $P_{i \leftarrow j}$.
Evaluate $Q_{i \leftarrow j}$.

Swapping integers

1. Imagine that we are trying to write a program that swaps the values of x and y without using any additional variables. We are not sure what the program should look like, and try three different suggestions:

   $$\texttt{x} = \texttt{x} - \texttt{y};\ \texttt{y} = \texttt{x} + \texttt{y};\ \texttt{x} = \texttt{y} - \texttt{x}$$

   $$\texttt{x} = \texttt{y} - \texttt{x};\ \texttt{y} = \texttt{y} - \texttt{x};\ \texttt{x} = \texttt{x} - \texttt{y}$$

   $$\texttt{x} = \texttt{x} + \texttt{y};\ \texttt{y} = \texttt{x} - \texttt{y};\ \texttt{x} = \texttt{x} - \texttt{y}$$

   Use Programming Logic (PL) to find out which suggestion(s) that actually swaps the values of x and y. Given the precondition $\{P : x == x_0 \wedge y == y_0\}$, where x_0 and y_0 are logical variables. Find out if $\{Q : y == x_0 \wedge x == y_0\}$ holds upon termination in each of the three cases.

2. Given the followng program S:

   ```
   S: if (x > y) { x = x + y; y = x - y; x = x - y}
   ```

   Prove that $\{\textbf{true}\}$ S $\{x \leq y\}$ is a theorem in PL.

If-else rule

A reasoning rule for **if-else** statements can be given as follows:

$$\frac{\{P \wedge B\}\ S_1\ \{Q\} \qquad \{P \wedge \neg B\}\ S_2\ \{Q\}}{\{P\}\ \textbf{if}\ (B)\ S_1\ \textbf{else}\ S_2\ \{Q\}}$$

Compare this rule to the reasoning rule for if statements given in figure 2.3.

Use the rule above to prove the following triple:

$$\begin{array}{l} \{x > 0 \wedge y > 0 \wedge x \neq y \wedge x + y == a\} \\ \textbf{if}\ (\texttt{x} < \texttt{y})\ \ \texttt{y} = \texttt{y} - \texttt{x}\ \ \textbf{else}\ \ \texttt{x} = \texttt{x} - \texttt{y} \\ \{x > 0 \wedge y > 0 \wedge x + y < a\} \end{array}$$

Exercises from the book

2.20