Beamspace Adaptive Beamforming and the GSC

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- The MVDR beamformer: performance and behavior.
- Generalized Sidelobe Canceller reformulation.
- Implementation of the GSC.
- Beamspace interpretation of the GSC.
- Reduced complexity of beamspace MVDR.

Model: Signal, (spatially white) noise, and interference for M-element array

$$\vec{x} = A\vec{d} + \vec{n}, E\left\{\vec{x}\vec{x}^H\right\} = \mathbf{R}_s + \mathbf{R}_n, \mathbf{R}_s = |A|^2\,\vec{d}\vec{d}^H, \mathbf{R}_n = \mathbf{R}_i + \sigma_w^2\mathbf{I} \qquad (1)$$

For spatially white noise only, the DAS beamformer is optimal (in the sense of minimum noise power in output):

$$y_{DAS} = \vec{w}_{DAS}^H\vec{x} \text{ for } \vec{w}_{DAS} = \frac{1}{M}\vec{d} \qquad (2)$$

For spatially non-white interference, the Minimum Variance beamformer minimizes interference-plus-noise power in the beamformer output:

$$\vec{w}_{mv} = \text{argmin}_{\vec{w}} E\left\{\left|\vec{w}^H\vec{x}\right|^2\right\} = \text{argmin}_{\vec{w}}\vec{w}^H\mathbf{R}\vec{w} \qquad (3)$$

In other words:

$$E\left\{\left|\vec{w}_{MV}^H\vec{n}\right|^2\right\} \leq E\left\{\left|\vec{w}^H\vec{n}\right|^2\right\} \text{ for all weight vectors } \vec{w} \qquad (4)$$

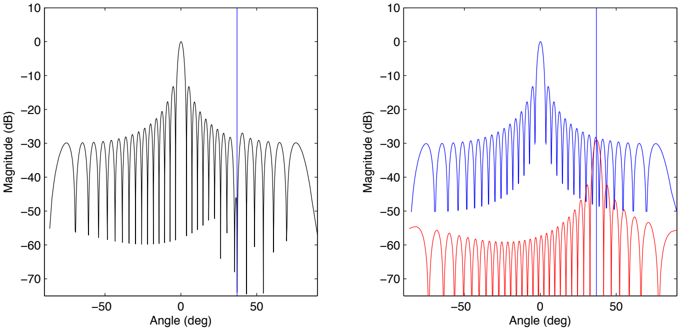
Note: This is actually *minimum power*, not *minimum variance*. Subtle difference, theoretical equivalence. Exchange $\mathbf{R}$ for $\mathbf{R}_n$...

Repetition: MVDR Beamforming 2

MV weight vector for the case of one single interfering source with power σ_i^2 and propagation vector $\vec{d}_i$:

$$\vec{w}_{MV} = \frac{\Lambda M}{\sigma_w^2}\left(\vec{w}_{das,s} - \rho_{si}\frac{M\sigma_i^2}{\sigma_w^2 + M\sigma_i^2}\vec{w}_{das,i}\right) \text{ for } \vec{w}_{das,s} = \frac{\vec{v}_s}{M}, \vec{w}_{das,i} = \frac{\vec{v}_i}{M} \quad (5)$$

- **Interpretation:** DAS beamformer steered towards signal minus (scaled) DAS beamformer steered towards interference.
- Scaling depends on INR, i.e. $\frac{\sigma_i^2}{\sigma_w^2}$



Generalized Sidelobe Canceller

- Constrained minimization is sometimes difficult to implement and analyze.
- *However:* MVDR can be reformulated as unconstrained minimization.
- First suggested by Griffiths and Jim (1982) as an alternative implementation of Frosts Linearly Constrained MV (LCMV) beamformer (1972).
- This implementation is usually referred to as the *Griffiths-Jim beamformer* or the *GSC*.

Given a matrix $B \in \mathbb{C}^{M,M-1}$ such that $\vec{d}^H\mathbf{B} = \vec{0}$. Then the constrained optimization problem:

$$\min_{\vec{w}} \vec{w}^H\mathbf{R}\vec{w} \text{ s.t. } \vec{w}^H\vec{d} = 1 \tag{6}$$

is identical to the unconstrained optimization problem:

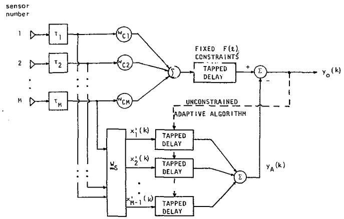
$$\min_{\vec{\beta}}(\frac{1}{M}\vec{d} - \mathbf{B}\vec{\beta})^H\mathbf{R}(\frac{1}{M}\vec{d} - \mathbf{B}\vec{\beta}) \tag{7}$$

with solution:

$$\vec{\beta} = \left(\mathbf{B}^H\mathbf{R}\mathbf{B}\right)^{-1}\mathbf{B}^H\mathbf{R}\frac{\vec{d}}{M} \tag{8}$$

GSC Implementation

- From now on: Assume $\vec{d} = \vec{1}$, i.e. signal arriving from broadside.
- Can be implemented as a transversal adaptive filter (using e.g. LMS or RLS algorithm).
- Looks like a *Wiener filter*, in that it subtracts adaptively filtered noise from desired signal...
- ...however, it does not produce MMSE output.
- **Note:** Sensitive to signal-interference correlation, just like MVDR.



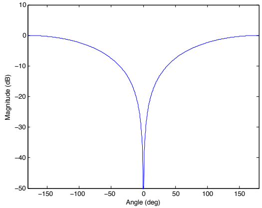
Blocking matrix **B** suggested by Griffiths and Jim:

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & 0 & \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \tag{9}$$

What signals are processed by the lower branch of the GSC?

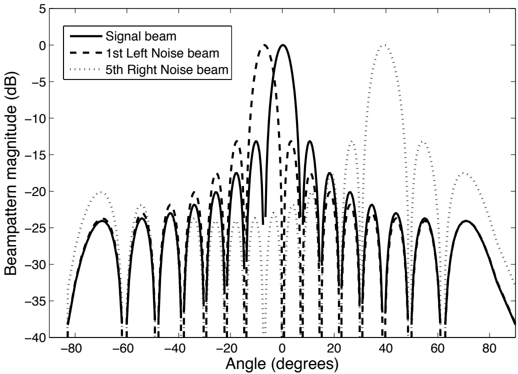
$$x'_m = x_m - x_{m+1} \text{ for } m = 0, 1, \cdots, M-2 \tag{10}$$

Two-element endfire "beams". Good for broadband; broadside nulling for all frequencies.



$$\mathbf{D} = \frac{1}{M}\left[\vec{d}_0, \cdots, \vec{d}_{M-1}\right], \left[\vec{d}_m\right]_n = e^{j\frac{2\pi m}{M}n} \tag{11}$$

- Special case: ULA with DFT matrix $\mathbf{B} = \mathbf{D}$ without first column ($\vec{d}_0 = \vec{w}_{DAS}$ for broadside arrival).
- Invertible transformation: $\mathbf{D}^H\mathbf{D} = \mathbf{D}\mathbf{D}^H = \frac{1}{M}\mathbf{I}$.



The constrained optimization problem becomes:

$$\min_{\vec{w}} E\left\{\left|(\vec{w}^H\mathbf{D}^H)(\mathbf{D}\vec{x})\right|^2\right\} \text{ s.t. } \vec{w}^H\vec{1} = 1$$

$$\Rightarrow \min_{\vec{w}_{BS}} E\left\{\left|\vec{w}_{BS}^H\vec{x}_{BS}\right|^2\right\} = \vec{w}_{BS}^H\mathbf{R}_{BS}\vec{w}_{BS} \text{ s.t. } \vec{w}_{BS}\vec{e}_0 = 1 \quad (12)$$

Beamspace solution:

$$\vec{w}_{BS} = \frac{\mathbf{R}_{BS}^{-1}\vec{e}_0}{\vec{e}_0^H\mathbf{R}_{BS}^{-1}\vec{e}_0} \quad (13)$$

GSC Interpretation: Beamspace

Beamspace beamformer output:

$$y_{BS} = x_{BS,0} + \sum_{m=1}^{M-1} w^*_{BS,m} x_{BS,m} \tag{14}$$

- Interpretation is more obvious: DAS beamformer minus weighted set of other DAS beamformers.
- Note that there is only information about the signal in $x_{BS,0}$.
- Certain beams $x_{BS,m}$ contain more information about interference than others.
- **Idea:** Remove those $x_{BS,m}$ that are expected to contain little or no information about interference.
- **Result:** $\vec{x}_{BS}$ becomes smaller, i.e. $\mathbf{R}_{BS}$ becomes smaller and easier to invert.
- Inversion of $\mathbf{R}$ is $O(M^3)$.
- Reduced-dimension beamspace presents a way of using a priori knowledge in the adaptive beamformer...
- ...but is this realistic knowledge?

Try $\mathbf{B} = \vec{d}_m$, base weights on *single-snapshot noise cov. matrix* $\mathbf{R}_n = \vec{n}\vec{n}^H$:

$$\beta = \left(\vec{d}_m^H \mathbf{R}_n \vec{d}_m\right) \vec{d}_m^H \mathbf{R}_n \vec{w}_{das} = \frac{y_{p,d} y_{p,das}^*}{|y_{p,d}|^2 + \frac{\sigma_w^2}{M}} \tag{15}$$

Yields beamspace MV weight vector:

$$\vec{w}_{bs} = \vec{w}_{das} - \beta \vec{d}_m \tag{16}$$

Yields interference in output:

$$y_{p,bs} = y_{p,das}\left(1 - \frac{|y_{p,d}|^2}{|y_{p,d}|^2 + \frac{\sigma_w^2}{M}}\right) \tag{17}$$

- Is the choice of $\vec{d}_m$ arbitrary?
- What is the impact on the white noise gain?
- What is the impact of spatially white noise?

- **Special case:** Adaptive Sidelobe Reduction (from Synthetic Aperture Radar).
- **Summary:** Only use $x_{BS,m}$ for $m = 0, 1, M-1$ and set $w_{BS,0} = 1, w_{BS,1} = w_{BS,M-1} = \alpha$.
- Complexity reduced from determining M weights to 1 weight.
- Additionally, solution is on the form:

$$[\vec{w}_{ASR}]_m = 1 + \alpha \cos\left(\frac{2\pi m}{M}\right) \tag{18}$$

which corresponds to a known family of windows (including Hamming and Hann).

- GSC interpretation of MVDR beamformer yields *unconstrained optimization problem*.
- Unconstrained optimization problems are often easier to analyze and implement.
- Beamspace interpretation of GSC can give reduced complexity.
- **Example:** Ultrasound imaging. Reduction from 64-dimensional element space to 3-dimensional beamspace with similar performance. In general: $O(M) \rightarrow O(3)$.