# Adaptive beamforming

Sven Peter Näsholm
Department of Informatics, University of Oslo
Spring semester 2012
svenpn@ifi.uio.no
Office phone number: +47 22840068

## Slide 2: Chapter 7: Adaptive array processing

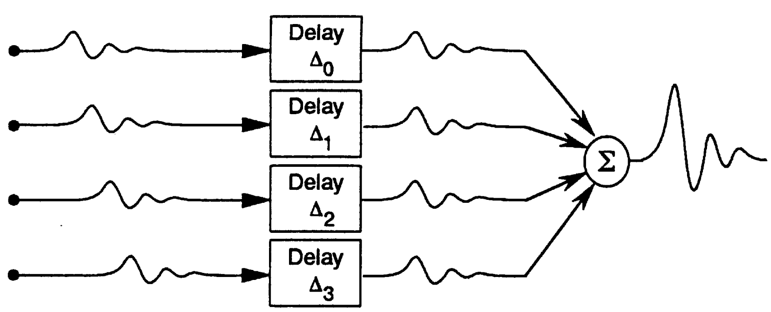
Minimum variance beamforming

- Generalized sidelobe canceler
- Signal coherence
- Spatial smoothing to de-correlate sources

Eigenanalysis methods

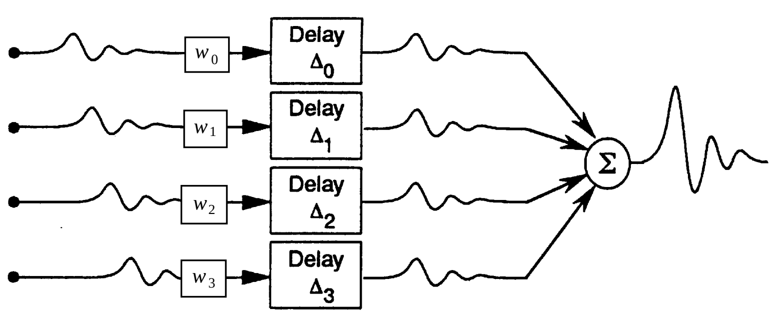
- Signal/noise subspaces
- Eigenvector method
- MUSIC

## Slide 3: Delay-and-sum




*stacking* $\triangleq$ adjustment of $\Delta_0 \ldots \Delta_{M-1}$

## Slide 4: Delay-and-sum, *continued*



- $z(t) = \sum_{m=0}^{M-1} w_m e^{-j\omega\Delta_m} y_m(t)$
- $w_m$: weight on signal $m$ $\Rightarrow$ *shading* = *apodization*
- *Conventional D-A-S:* <u>$\mathbf{w}$ *independent of recorded signal data*</u>

# Slide 5: Delay-and-sum on vector form

- Monochromatic source: $y_m(t) = e^{j(\omega t - \vec{k}^\circ \cdot \vec{x}_m)}$.
- Delayed signal: $y_m(t - \Delta_m) = y_m(t) e^{-j\omega\Delta_m}$

Define:

$$\text{Weights: } \mathbf{w} \triangleq \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{bmatrix} \quad \begin{matrix}\text{Received}\\ \text{signals:}\end{matrix} \; \widetilde{\mathbf{Y}(t)} \triangleq \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{M-1}(t) \end{bmatrix} \quad \text{Delayed } \widetilde{\mathbf{Y}(t)} : \mathbf{Y}(t) \triangleq \begin{bmatrix} y_0(t)e^{-j\omega\Delta_0} \\ y_1(t)e^{-j\omega\Delta_1} \\ \vdots \\ y_{M-1}(t)e^{-j\omega\Delta_{M-1}} \end{bmatrix}$$

- Beamf. output: $z(t) = \sum_{m=0}^{M-1} w_m e^{-j\omega\Delta_m} y_m(t) = \mathbf{w}^{\mathrm{H}} \mathbf{Y}(t)$

□

- Power of $z(t)$:

$$P(z(t)) \triangleq \mathrm{E}\{|z(t)|^2\} = \mathrm{E}\{(\mathbf{w}^{\mathrm{H}}\mathbf{Y})(\mathbf{w}^{\mathrm{H}}\mathbf{Y})^{\mathrm{H}}\} = \mathrm{E}\{\mathbf{w}^{\mathrm{H}}\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\mathbf{w}\}$$
$$= \mathbf{w}^{\mathrm{H}}\, \mathrm{E}\{\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\}\, \mathbf{w} = \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w}$$

□

# Slide 5: Delay-and-sum on vector form, *continued*

Spatial covariance matrix in case of steering

$$\mathbf{R} \triangleq \mathrm{E}\{\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\}$$
$$= \begin{bmatrix} \mathrm{E}\{y_0(t)e^{-j\omega\Delta_0} y_0^*(t) e^{j\omega\Delta_0}\} & \mathrm{E}\{y_0(t)e^{-j\omega\Delta_0} y_1^*(t) e^{j\omega\Delta_1}\} & \cdots & \mathrm{E}\{y_0(t)e^{-j\omega\Delta_0} y_{M-1}^*(t) e^{j\omega\Delta_{M-1]}}\} \\ \mathrm{E}\{y_1(t)e^{-j\omega\Delta_1} y_0^*(t) e^{j\omega\Delta_0}\} & \mathrm{E}\{y_1(t)e^{-j\omega\Delta_1} y_1^*(t) e^{j\omega\Delta_1}\} & \cdots & \mathrm{E}\{y_1(t)e^{-j\omega\Delta_1} y_{M-1}^*(t) e^{j\omega\Delta_{M-1]}}\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}\{y_{M-1}(t)e^{-j\omega\Delta_{M-1}} y_0^*(t) e^{j\omega\Delta_0}\} & \mathrm{E}\{y_{M-1}(t)e^{-j\omega\Delta_{M-1}} y_1^*(t) e^{j\omega\Delta_1}\} & \cdots & \mathrm{E}\{y_{M-1}(t)e^{-j\omega\Delta_{M-1}} y_{M-1}^*(t) e^{j\omega\Delta_{M-1}}\} \end{bmatrix}$$

# Slide 6: Delay-and-sum on vector form, *continued*

Define:
*Steering vector*:

$$\mathbf{e} \triangleq \begin{bmatrix} e^{-j\vec{k}\cdot\vec{x}_0} \\ e^{-j\vec{k}\cdot\vec{x}_1} \\ \vdots \\ e^{-j\vec{k}\cdot\vec{x}_{M-1}} \end{bmatrix}$$

- $z(t) = \mathbf{w}^{\mathrm{H}}\mathbf{Y}$
- Power of $z(t)$:

$$P(\mathbf{e}) \triangleq \mathrm{E}\{|z(t)|^2\}(\mathbf{e}) = \mathbf{w}^{\mathrm{H}}\mathbf{R}(\mathbf{e})\mathbf{w} \xlongequal{\text{if } w_m = 1, \forall m} \mathbf{1}^{\mathrm{H}}\mathbf{R}(\mathbf{e})\,\mathbf{1}$$

# Slide 7: Delay-and-sum on vector form, *continued*

About $\mathbf{e}$:

- *Steering vector*

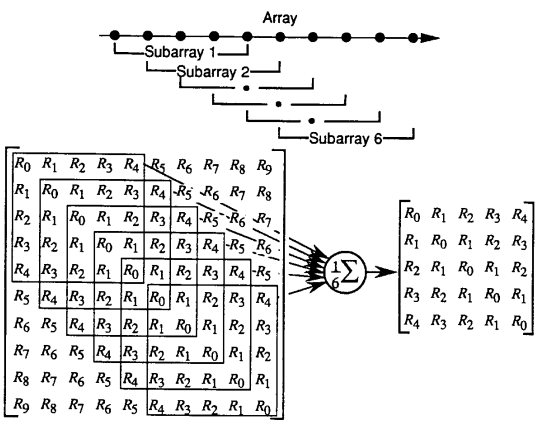
$$\mathbf{e} \triangleq \begin{bmatrix} e^{-j\vec{k}\cdot\vec{x}_0} \\ e^{-j\vec{k}\cdot\vec{x}_1} \\ \vdots \\ e^{-j\vec{k}\cdot\vec{x}_{M-1}} \end{bmatrix}$$

- Contains delays to focus in specific direction
- Represents unit amplitude signal, propagating in $\vec{k}$ direction

*In conventional D-A-S:* $\mathbf{w}$ *independent of received signal data*

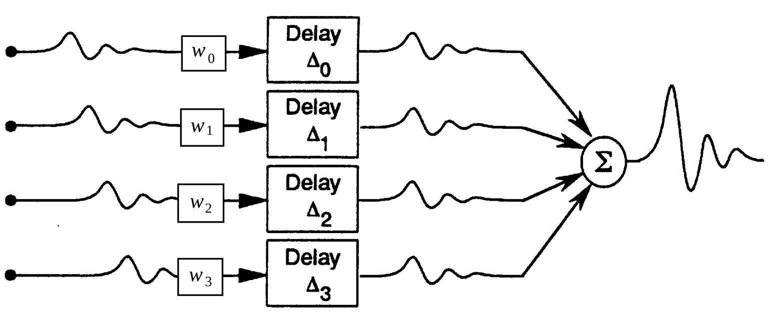
# Slide 8: Estimation of spatial covariance matrix

- Averaging in time
- Averaging in space = spatial smoothing = subarray averaging




# Slide 9: Minimum variance beamforming

## Assume narrow-band signals




- Adaptive method. [Latin: *adaptare* "to fit to"]
- Allow $w_m$ to also be complex and/or negative
- Sensor weights defined *not only as function of problem geometry, but also as function of received signals*

# Slide 10: Minimum variance beamforming, *continued*

- "Minimum variance" = "Capon" = "Maximum likelihood" (beamforming)
- Constrained optimization problem
- New steering direction $\mathbf{e} \Rightarrow$ new calculation of element weights $w_m$

Minimum-variance key equations

$$\mathbf{w} \triangleq \begin{bmatrix} w_0 \\ \vdots \\ w_{M-1} \end{bmatrix} \Rightarrow \boxed{\begin{array}{c} \min_{\mathbf{w}} \left(\mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w}\right), \\ \text{constraint:} \\ \mathbf{w}^{\mathrm{H}}\mathbf{e} = 1 \end{array}} \Leftrightarrow \boxed{\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{e}}{\mathbf{e}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{e}}} \Rightarrow \boxed{P(\mathbf{e}) = \frac{1}{\mathbf{e}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{e}}}$$

- [beampattern] $\neq$ [steered response]

# Slide 11: Minimum variance beamforming, *continued*

## Example

Uniform linear array, $M = 10$

Comparing:

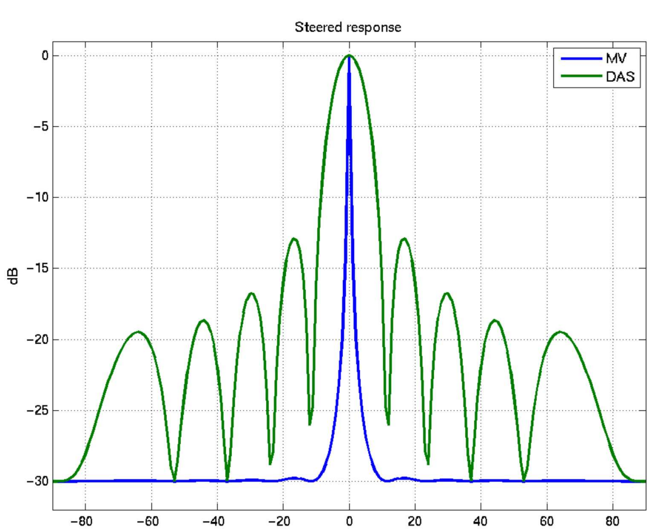
- Delay-and-sum
- Minimum-variance

Investigating:

- Steered responses
- Beampatterns

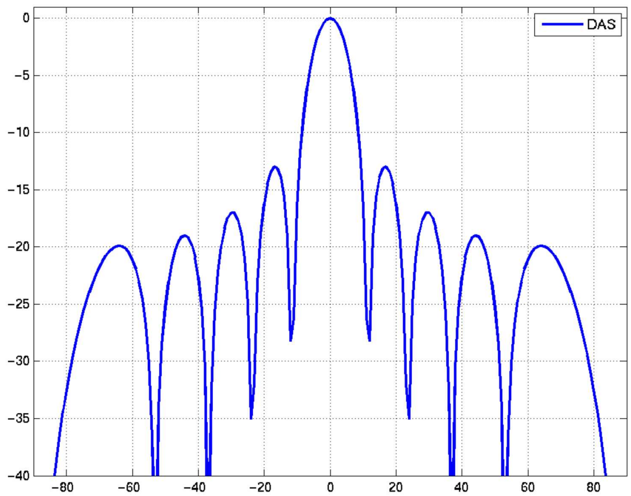
# Slide 12: Comparison: delay-and-sum / minimum-variance

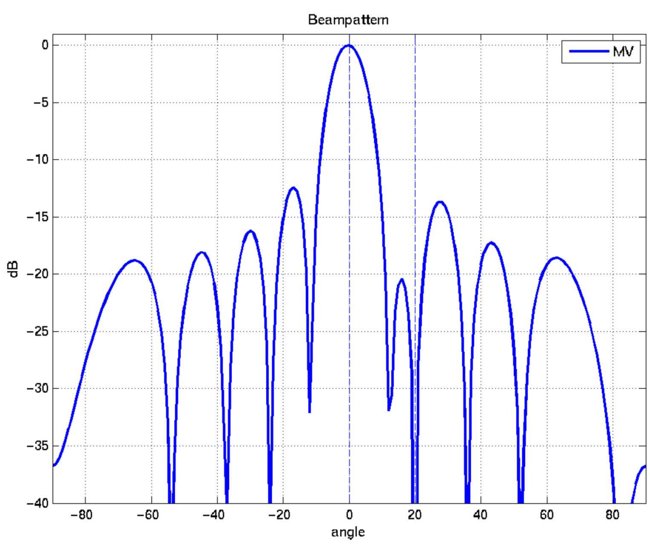
## 1 source

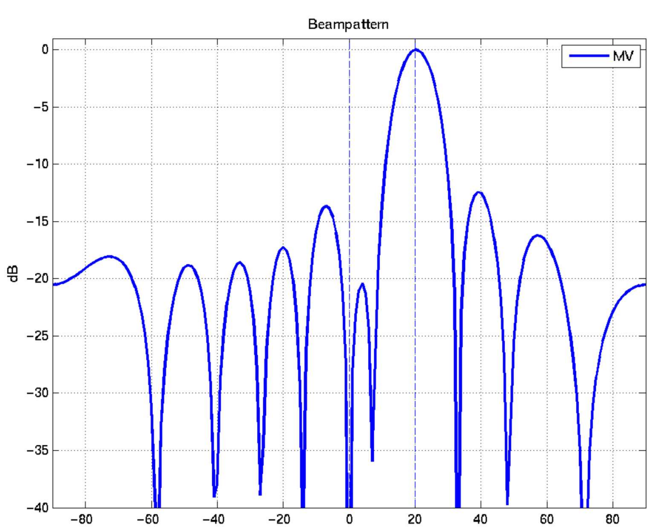


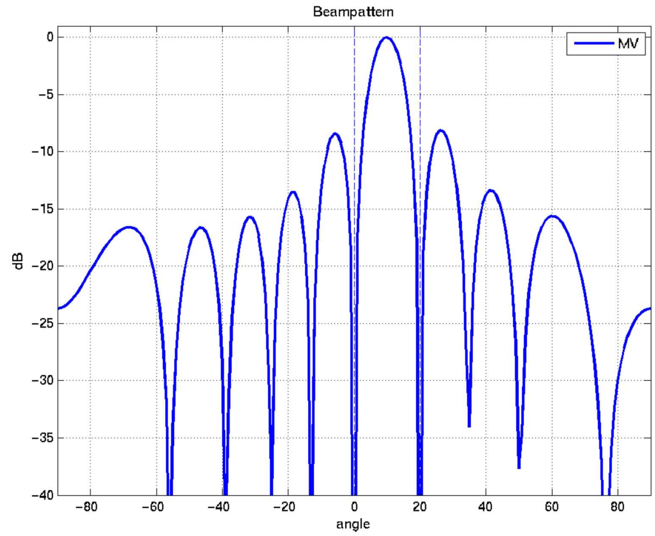
# Slide 13: Comparison: delay-and-sum / minimum-variance

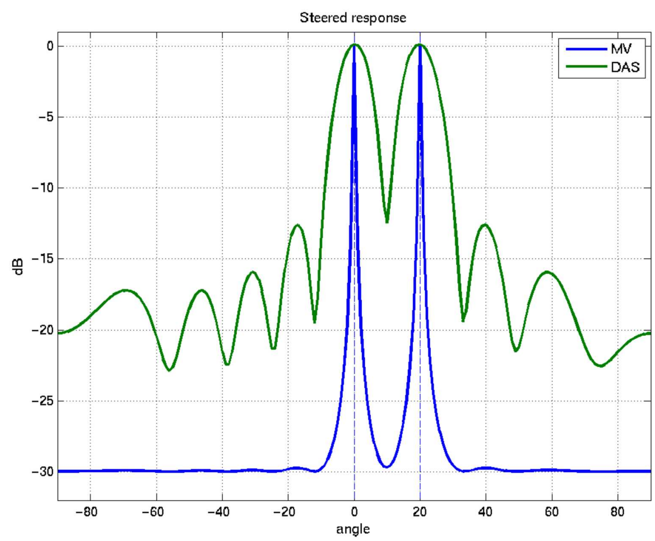
2 sources



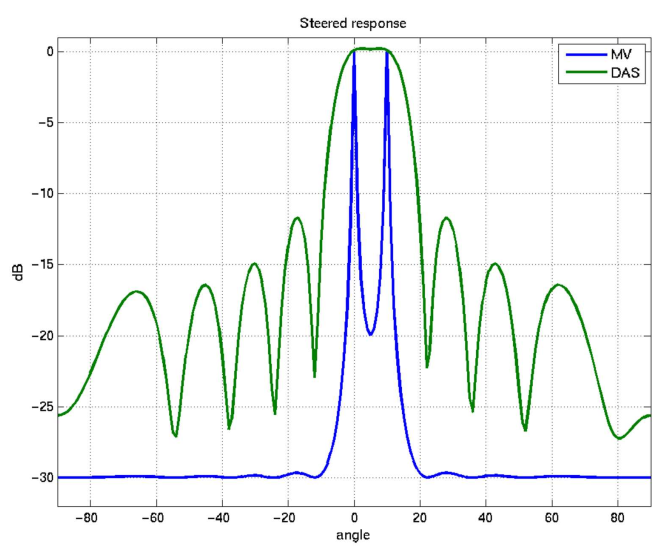





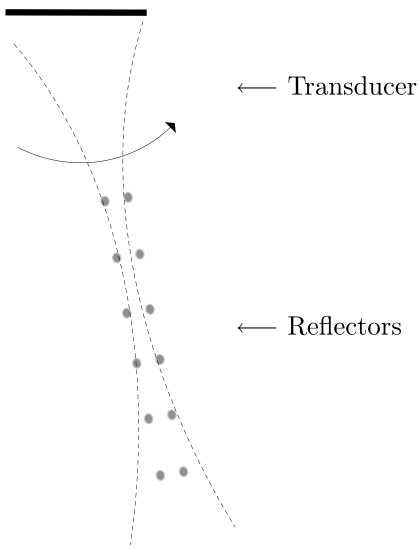

## Slide 14: Comparison: delay-and-sum / minimum-variance

2 closely spaced sources



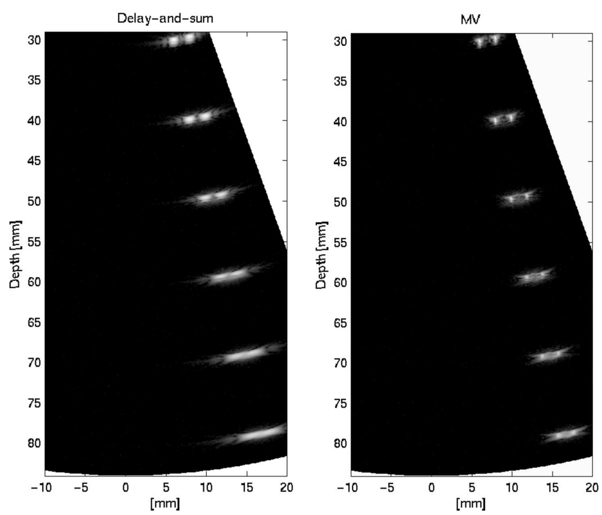
# Slide 15: Example: Ultrasound imaging

## Experimental setup



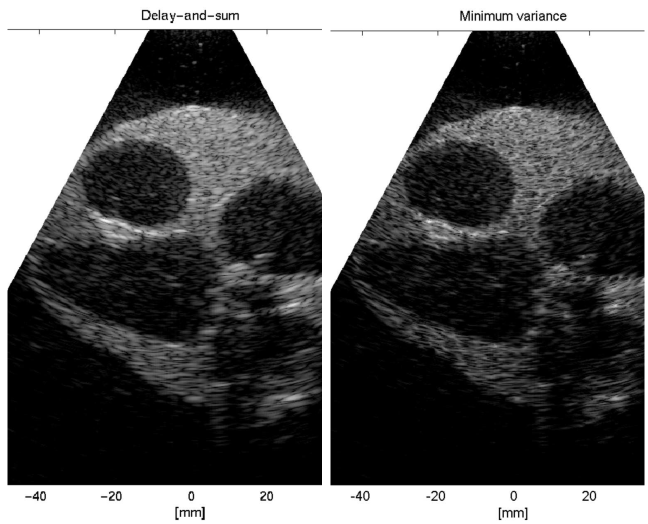

# Slide 16: Example: Ultrasound imaging, *continued*

## Resulting images

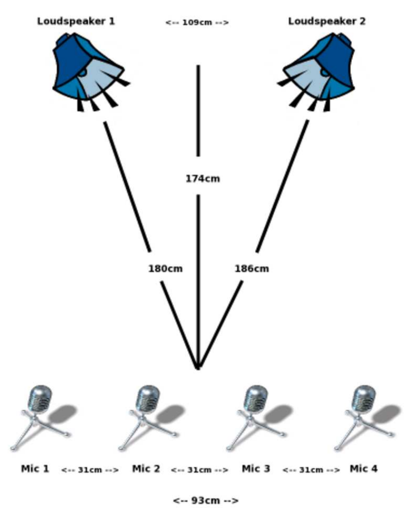


## Slide 17: Example: Ultrasound imaging, *continued*

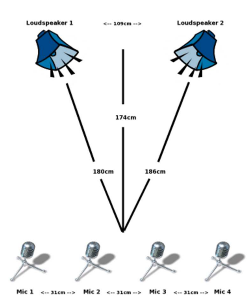
Experimental data, heart phantom




## Slide 18: Sound example: Microphone array




## Slide 19: Sound example, *continued*



- Single microphone `mix.wav`
- Delay-and-sum `gaute_DAS.wav`

- Minimum-variance `gaute_adaptive.wav`

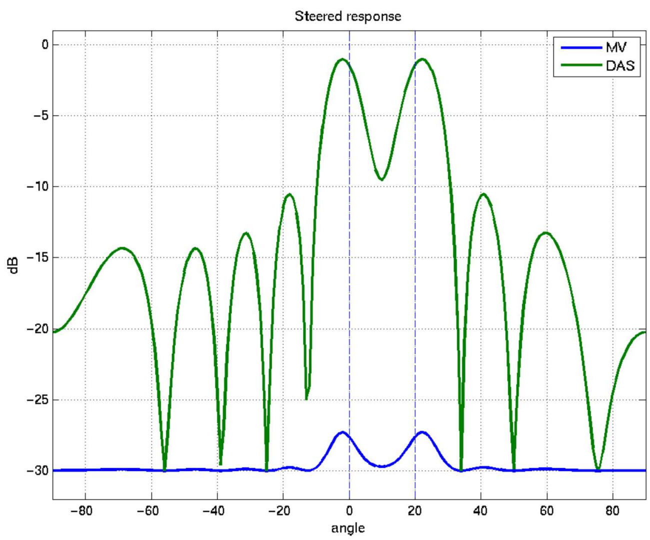
## Slide 20: Minimum-variance beamformer:

Uncorrelated signals required for optimum functioning

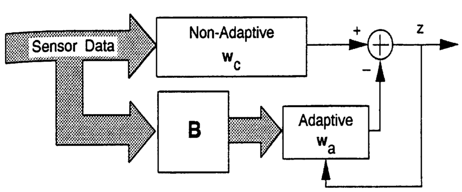
- Correlated signals may give signal cancellation
- Although constraint is fulfilled: signal propagating in the "look-direction" may be canceled by a correlated interferer

## Slide 21: Minimum-variance / D-A-S example

2 correlated sources




## Slide 22: Generalized sidelobe canceler




- Conventional D-A-S weights $\mathbf{w}_c$: steering in assumed propagation direction
- Adaptive/canceler portion: removal of other signals
- Blocking matrix $\mathbf{B}$, blocks signal from assumed propagation direction from coming into "canceler part"
- $\mathbf{w}_a$: adaptive weights to emphasize what is to be removed
- Minimize total power. Unconstrained minimization:

$$\min_{\mathbf{w}_a} \left(\mathbf{w}_c - \mathbf{B}^{\mathrm{H}}\mathbf{w}_a\right)^{\mathrm{H}} \mathbf{R} \left(\mathbf{w}_c - \mathbf{B}^{\mathrm{H}}\mathbf{w}_a\right) \Rightarrow \mathbf{w}_a = \left(\mathbf{B}\mathbf{R}\mathbf{B}^{\mathrm{H}}\right)^{-1} \mathbf{B}\mathbf{R}\mathbf{w}_c$$

- Full details: D&J pp. 369–371

# Slide 23: Eigenanalysis and Fourier analysis

(Chapter 7.3.1)

Eigenvalues / Eigenvectors of $\mathbf{R}$

$$\mathbf{R}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

$$\left.\begin{array}{lll} \mathbf{R}\ \underline{\text{Hermitian}}\ \text{(self-adjoint)} & \Leftrightarrow & \mathbf{R} = \mathbf{R}^{\mathrm{H}} \\ \mathbf{R}\ \underline{\text{Positive semidefinite}} & \Leftrightarrow & \mathbf{x}^{\mathrm{H}}\mathbf{R}\mathbf{x} \geq 0,\ \forall \mathbf{x} \neq \mathbf{0} \end{array}\right\} \Rightarrow$$

- Eigenvalues real & positive: $\lambda_i \geq 0,\ \forall i$
- Eigenvectors $\mathbf{v}_i$: orthogonal

$$\Rightarrow \text{For } \mathbf{V} \triangleq \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_M \\ | & | & & | \end{bmatrix}:\ \mathbf{V} \text{ is unitary matrix}$$

$$\Leftrightarrow \mathbf{V}^{\mathrm{H}}\mathbf{V} = \mathbf{I} \Leftrightarrow \boxed{\mathbf{V}^{-1} = \mathbf{V}^{\mathrm{H}}}$$

□

# Slide 24: Eigenvalue decomposition

□

Eigendecomposition of $\mathbf{R}$ & $\mathbf{R}^{-1}$: *The spectral theorem*

- $\mathbf{R} = \sum_{i=1}^{M} \lambda_i \mathbf{v}_i \mathbf{v}_i^{\mathrm{H}} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathrm{H}}$
- $\mathbf{R}^{-1} = \sum_{i=1}^{M} \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^{\mathrm{H}} = \mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{V}^{\mathrm{H}}$
- *Note above:* Same eigenvectors, but inverse eigenvalues

# Slide 25: [Signal + noise] & [noise] subspaces

$$\mathbf{R} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathrm{H}} = \underbrace{\mathbf{V}_{\mathrm{s}}\mathbf{\Lambda}_{\mathrm{s}}\mathbf{V}_{\mathrm{s}}^{\mathrm{H}}}_{\substack{\text{from} \\ \text{signal}}} + \underbrace{\mathbf{V}_{\mathrm{n}}\mathbf{\Lambda}_{\mathrm{n}}\mathbf{V}_{\mathrm{n}}^{\mathrm{H}}}_{\substack{\text{from} \\ \text{noise}}}$$

□

# Slide 26: Beamformer output powers

□

Minimum variance, $\hat{\mathbf{R}} = \mathrm{E}\{\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\}$: "full $\mathbf{R}$ estimate"

$$P_{\mathrm{MV}}(\mathbf{e}) = \frac{1}{\underbrace{\mathbf{e}^{\mathrm{H}}\mathbf{V}_{\mathrm{s}}\mathbf{\Lambda}_{\mathrm{s}}^{-1}\mathbf{V}_{\mathrm{s}}^{\mathrm{H}}\mathbf{e}}_{\substack{\text{signal-plus-noise}\\ \text{subspace}}} + \underbrace{\mathbf{e}^{\mathrm{H}}\mathbf{V}_{\mathrm{n}}\mathbf{\Lambda}_{\mathrm{n}}^{-1}\mathbf{V}_{\mathrm{n}}^{\mathrm{H}}\mathbf{e}}_{\substack{\text{noise}\\ \text{subspace}}}} = \frac{1}{\sum_{i=1}^{N_s} \frac{1}{\lambda_i}\left|\mathbf{e}^{\mathrm{H}}\mathbf{v}_i\right|^2 + \sum_{i=N_s+1}^{M} \frac{1}{\lambda_i}\left|\mathbf{e}^{\mathrm{H}}\mathbf{v}_i\right|^2}$$

Eigenvector method, $\hat{\mathbf{R}} = \sum_{i=N_s+1}^{M} \lambda_i \mathbf{v}_i \mathbf{v}_i^{\mathrm{H}}$: "noise-only $\mathbf{R}$ estimate"

$$P_{\mathrm{EV}}(\mathbf{e}) = \frac{1}{\mathbf{e}^{\mathrm{H}}\mathbf{V}_{\mathrm{n}}\mathbf{\Lambda}_{\mathrm{n}}^{-1}\mathbf{V}_{\mathrm{n}}^{\mathrm{H}}\mathbf{e}} = \frac{1}{\sum_{i=N_s+1}^{M} \frac{1}{\lambda_i}\left|\mathbf{e}^{\mathrm{H}}\mathbf{v}_i\right|^2}$$

MUSIC, $\hat{\mathbf{R}} = \sum_{i=N_s+1}^{M} \mathbf{v}_i \mathbf{v}_i^{\mathrm{H}}$: "normalized noise-only $\mathbf{R}$ estimate"

$$P_{\mathrm{MUSIC}}(\mathbf{e}) = \frac{1}{\mathbf{e}^{\mathrm{H}}\mathbf{V}_{\mathrm{n}}\mathbf{V}_{\mathrm{n}}^{\mathrm{H}}\mathbf{e}} = \frac{1}{\sum_{i=N_s+1}^{M} \left|\mathbf{e}^{\mathrm{H}}\mathbf{v}_i\right|^2}$$

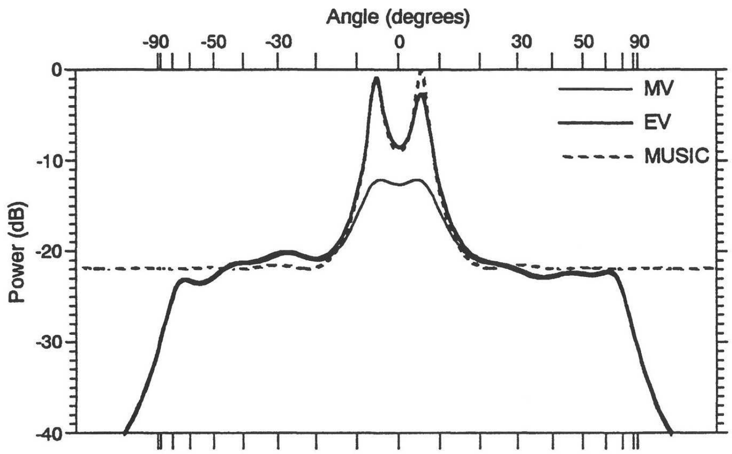
## Slide 27: [Signal + noise], & [noise] subspaces, *continued*

- $\mathbf{R}$ always Hermitian $\Rightarrow$ $\forall$ eigenvectors orthogonal
- $N_s$ largest eigenvectors: span the [signal+noise subspace]
- $M - N_s$ smallest span the [noise subspace]
- Signal steering vectors $\in$ [signal+noise subspace], $\perp$ [noise subspace]
- [Largest eigenvectors] $\neq$ [signal vectors] However: signal vectors are linear combinations of [largest eigenvectors]

## Slide 28: EV / MUSIC

- (Inverse of) projection of possible steering vectors onto the noise subspace
- MUSIC: Dropping $\lambda_i$ in $\hat{\mathbf{R}}$ $\Leftrightarrow$ noise whitening
- EV & MUSIC: peaks at Directions-Of-Arrival (DOAs)
  - No amplitude preservation
  - Require knowledge of # signals, $N_s$
  - How to estimate $N_s$?

## Slide 29: MV / EV / MUSIC comparison




## Slide 30: Situation with signal coherence

$$\mathbf{R} = \mathbf{SCS}^{\mathrm{H}} + \mathbf{K}_{\mathrm{n}}$$

- signal coherence $\Rightarrow$ rank deficient signal matrix $\mathbf{SCS}^{\mathrm{H}}$
- For minimum-variance: May cause signal cancellation
- Gives signal+noise-subspace consisting of linear combinations of signal vectors
- EV & MUSIC: may fail to produce peaks at DOA locations
- Want to reduce cross-correlation terms in $\mathbf{C}$.
- Cure: Spatial smoothing (subarray averaging)