Taylor-polynom av grad n om punktet a

$$T_n f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Restledd

$$R_n f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

der c er et tall mellom a og x

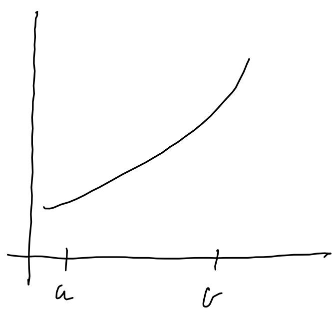
$$f(x) = \underbrace{T_n f(x)}_{\text{Tilnærming til } f(x)} + \underbrace{R_n f(x)}_{\text{Feil / restledd}}$$

$$\left| f(x) - T_n f(x) \right| = \left| R_n f(x) \right|$$

Vi må begrense $\left| f^{(n+1)}(c) \right|$ for $c \in (a, x)$

(evt. $c \in (x, a)$ hvis $x < a$)

"Tre" tilfeller

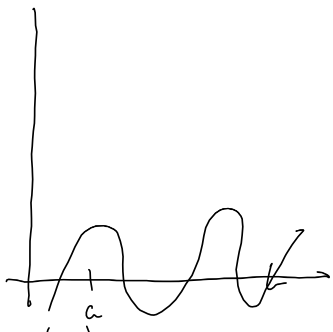


$f^{(n+1)}(c)$ er økende

$\left| f^{(n+1)}(c) \right| \leq \left| f(b) \right|$

$f^{(n+1)}(c)$ er avtagende

$\left| f^{(n+1)}(c) \right| \leq f(a)$



$f^{(n+1)}(c)$ er sin eller cos

da er $\left| f^{(n+1)}(c) \right| \leq 1$

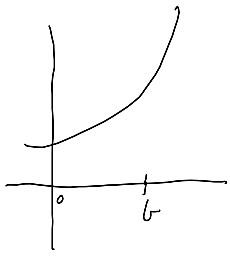
11.2.1 $\quad f(x) = e^x$ $\qquad$ Grad 4 om punktet $a = 0$

$$f(x) = e^x \qquad f(0) = 1$$
$$f'(x) = e^x \qquad f'(0) = 1$$
$$f''(x) = e^x \qquad f''(0) = 1$$
$$f'''(x) = e^x \qquad f'''(0) = 1$$
$$f^{(4)}(x) = e^x \qquad f^{(4)}(0) = 1$$

$$T_4 f(x) = \frac{f(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Vis at $R_4 f(b) \le \frac{e^b}{120} b^5 \qquad b \geq 0$

$f^{(3)}(x) = e^x$



$$|R_4 f(b)| = \left|\frac{e^c}{5!}(b-0)^5\right| \le \frac{e^b}{120} b^5$$

<u>11.2.2</u> Grad 4 om punkte $a = 0$

$f(x) = \sin x$ $f(0) = 0$

$f'(x) = \cos x$ $f'(0) = 1$

$f''(x) = -\sin x$ $f''(0) = 0$

$f'''(x) = -\cos x$ $f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $f^{(4)}(0) = 0$

$$T_4 f(x) = \frac{1}{1!}(x-0) + \frac{-1}{3!}(x-0)^3$$

$$= x - \frac{1}{6}x^3$$

<u>Feil:</u>

$f^{(5)}(x) = \cos x$

$$|f(b) - T_4 f(b)| = |R_4 f(b)| = \left| \frac{f^{(5)}(c)}{5!}(b-0)^5 \right| \leq \frac{1}{120}|b|^5$$

11.2.15

a) Grad 2 om punktet $a = 0$

$f(x) = (1+x)^{\frac{1}{3}}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$ $\qquad f'(0) = \frac{1}{3}$

$f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}}$ $\qquad f''(0) = -\frac{2}{9}$

$$T_2 f(x) = 1 + \frac{1}{3}x - \frac{\frac{2}{9}}{2!}x^2$$

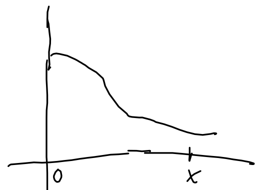
$$= 1 + \frac{1}{3} - \frac{1}{9}x^2$$

b) Vis at for $x \geq 0$ er $|R_2 f(x)| \leq \frac{5}{81}x^3$

$$f'''(x) = \frac{10}{27}(1+x)^{-\frac{8}{3}}$$

$$|R_2 f(x)| = \left|\frac{f'''(c)}{3!}x^3\right| = \left|\frac{5}{81}(1+c)^{-\frac{8}{3}}x^3\right| = \frac{5}{81}x^3\left|\frac{1}{(1+c)^{8/3}}\right|$$

$$\leq \frac{5}{81}x^3\left|\frac{1}{(1+0)^{8/3}}\right| = \frac{5}{81}x^3$$



$c \in (0, x)$

c) Finn $\sqrt[3]{1003}$ med 7 gjeldene desimaler

$$\sqrt[3]{1003} = \sqrt[3]{1000+3} = \sqrt[3]{1000(1+0,003)} = \sqrt[3]{10^3(1+0,003)}$$

$$= 10\sqrt[3]{1+0,003} = 10(1+0,003)^{1/3} = 10 \cdot f(0,003)$$

$$|10f(0,003) - 10T_2 f(0,003)| = |10 R_2 f(0,003)| \leq 10 \cdot \frac{5}{81}\left(\frac{3}{1000}\right)^3$$

$$\frac{\cancel{10}\cdot 5}{3^4}\cdot\frac{\cancel{3^3}}{10^9} = \frac{5}{3}\cdot 10^{-8}$$