Numerisk integrasjon

Midtpunktsmetoden (MM)

$$\int_a^b f(x)\,dx \approx I_{mid}(h) = h \sum_{i=1}^{n} f(x_{i-1/2})$$

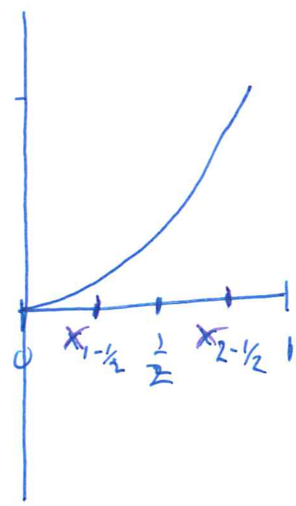
$$x_{i-1/2} = \frac{x_i + x_{i-1}}{2} = a + \left(i - \tfrac{1}{2}\right)h \qquad h = \frac{b-a}{n}$$

12.2.2

Bruk MM til å tilnærme

$$\int_0^1 x^2\,dx$$

med 2 delintervaller.



$$h = \frac{1-0}{2} = \frac{1}{2}$$

$$x_{1-1/2} = 0 + (1 - 1/2)h = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$x_{2-1/2} = 0 + (2 - 1/2)h = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}$$

$f(x) = x^2$

$f\left(\frac{1}{4}\right) = \frac{1}{16}$

$f\left(\frac{3}{4}\right) = \frac{9}{16}$

$$I_{mid}\left(\tfrac{1}{2}\right) = \frac{1}{2}\left(f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right)\right) = \frac{1}{2}\left(\frac{10}{16}\right) = \frac{5}{16}$$

~~1.2.1~~ 12.2.1

c) MM gir nøyaktig resultat for polynomer av grad 2.

Altså $p(x) = ax^2 + bx + c$. $a \neq 0$

~~Da~~ Usant

har vi $p''(x) = 2a$

Så feilen er

$$\left| \int_a^b p(x)\,dx - I_{mid}(h) \right| \leq (b-a)\frac{h^2}{24} 2a \neq 0$$

d) Den globale feilen i midtpunktsmetoden er en grad lavere enn den lokale

Sant!

Lokal feil

$$\frac{h^3}{24} M$$

Global feil

$$\frac{h^2}{24}(b-a) M$$

$$M = \max_{x \in [a,b]} |f''(x)|.$$

1.2.2.1 Sant eller usant?

a) Med MM er avrundingsfeil som følge av subtraksjon av to nesten like tall ~~ett~~ en hovedkilde til avrundingsfeil.

Usant.

$$\bar{I}_{mid}(h) = h \sum_{i=1}^{n} f(x_{i-1/2})$$

Vi kan anta at $f(x_{i-1/2}) \approx f(x_{i+1/2})$ så vi har bare addisjon.

b) MM gir ~~nøyaktig~~ nøyaktig svar for polynomer av grad 1. Altså for $p(x) = ax + b$

Sant.

Vi har at feilen er gitt ved

$$\left| \int_a^b p(x)\,dx - \bar{I}_{mid}(h) \right| \le (b-a)\frac{h^2}{24} \max_{x\in[a,b]} \left| \overset{0}{\overbrace{p''(x)}} \right| \le 0$$

men $p''(x) = 0$, ~~fordi~~ ~~gitt~~

12.2.1

e) For MM: Hvis vi reduserer h med en faktor 3 vil feilen reduseres med en faktor 9.

Sant

$$\text{Error}\left(I_{mid}(h)\right) \leq \frac{h^2}{24}(b-a)M$$

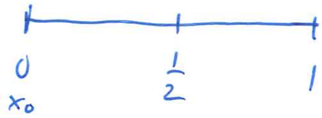
$$\text{Error}\left(I_{mid}\left(\frac{h}{3}\right)\right) \leq \frac{\frac{h^2}{3^2}}{24}(b-a)M \leq \frac{1}{9}\frac{h^2}{24}(b-a)M$$

$$= \frac{1}{9}\,\text{Error}\left(I_{mid}(h)\right)$$

$h = \frac{1-0}{2} = \frac{1}{2}$

12.3.2

$h = 2$



$$\int_0^1 x^2\,dx = \frac{0^2 + \left(\frac{1}{2}\right)^2}{2} h + \frac{\left(\frac{1}{2}\right)^2 + 1^2}{2} h$$

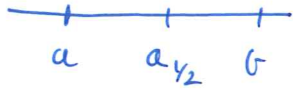
$$= \frac{\frac{1}{4}}{2} h + \frac{\frac{1}{4} + 1}{2} h$$

$$= \frac{1}{8} h + \frac{5}{8} h$$

$$= \frac{1}{16} + \frac{5}{16} = \frac{6}{16} = \underline{\underline{\frac{3}{8}}}$$

$h = \frac{1}{2}$

12.2.6



$a_{1/2} = \frac{a+b}{2}$

$c \in (a, x)$

$$T_1 f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

$$\int_a^b f(x) = \int_a^b f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2\,dx$$

$$= f(a)x\Big|_a^b + f'(a)\left(\tfrac{1}{2}x^2 - ax\right)\Big|_a^b + \int_a^b \frac{f''(c)}{2}(x-a)^2\,dx$$

$$= f(a)(b-a) + f'(a)\underbrace{\tfrac{1}{2}(b^2-a^2) - a(b-a)} + \underbrace{\int_a^b \frac{f''(c)}{2}(x-a)^2\,dx}$$

$\frac{1}{2}b^2 - \frac{1}{2}a^2 - ab + a^2$

$= \frac{1}{2}(b^2 - 2ab + a^2)$

$= \frac{1}{2}(b-a)^2$

Ved å følge samme utregning som i komp. får vi: $\le \frac{M}{24}(b-a)^3$

$M = \max_{x \in [a,b]} |f''(x)|$

$$\le \underbrace{f(a)(b-a)}_{\text{Ny formel}} + \underbrace{\overbrace{\frac{f'(a)(b-a)^2}{2}}^{\text{Det dominerende leddet}} + \frac{M}{24}(b-a)^3}_{\text{Ny feil.}}$$

Lokal feil blir nå av orden 2 og ikke orden 3.

Sant eller Usant

12.3.1

a) Trapesmetoden er vanligvis mer nøyaktig enn MM.

~~S~~ Usant

Vi har at $\quad Err(I_{mid}(h)) = \frac{M}{24}(b-a)h^2$

$$Err(I_{trap}(h)) = \frac{M}{12}(b-a)h^2 = 2\,\frac{M}{24}(b-a)h^2$$

$$= 2 \cdot Err(I_{mid}(h))$$

b) Siden hvert punkt i trapesmetoden blir evaluert to ganger, må vi evaluere funksjonen to ganger i hvert punkt.

Usant $\quad a = x_0 < x_1 < \cdots < x_n = b$

$$I_{trap}(h) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} h$$

$$= \frac{h}{2}\sum_{i=0}^{n-1} f(x_i) + \frac{h}{2}\sum_{i=0}^{n-1} f(x_{i+1})$$

$$= \frac{h}{2}\left(\cancel{f(x_0)+} \sum_{i=0}^{n-1} f(x_i) + \sum_{i=1}^{n} f(x_i)\right)$$

$$= \frac{h}{2}\left(f(x_0) + \left(\sum_{i=1}^{n-1} f(x_i)\right) + \left(\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right)$$

$$= \frac{h}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n} f(x_i)\right)$$