Oblig 2 er lagt ut! Frist 3. november

Ubestemte integraler

$\int f(x)\,dx = F(x) + C$ der $F(x)$ er en antideriverte til $f(x)$

Setning: Hvis f er kontinuerlig og g' er kontinuerlig, så er

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

der F er en antiderivert til f.

Bevis: Nok å vise at $F(g(x))$ er en antiderivert til $f(g(x))\,g'(x)$.

$$(F(g(x)))' = \underbrace{F'}_{f}(g(x))\,g'(x) = f(g(x))\,g'(x).$$

Huskeregel: $\int f(\underline{g(x)})\,\underline{g'(x)\,dx}$

$u = g(x)$

$\frac{du}{dx} = u' = g'(x)$

$du = g'(x)\,dx$

$= \int f(u)\,du = F(u) + C$

$= F(g(x)) + C.$

Eksempel: $\int x e^{x^2} dx$ $\quad u = x^2$, $\quad du = 2x\,dx$, $\quad x\,dx = \frac{1}{2}du$

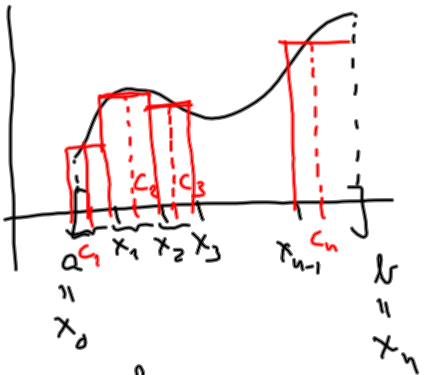
$$= \int e^u \tfrac{1}{2} du = \tfrac{1}{2}\int e^u du = \tfrac{1}{2} e^u + C = \underline{\tfrac{1}{2} e^{x^2} + C}$$

Riemann-summer

Partisjoner: $a = x_0 < x_1 < x_2 < \dots < x_n = b$

Utplukk: $c_1, c_2, \dots, c_n$ der $c_i \in [x_{i-1}, x_i]$

Riemann-sum $\displaystyle\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = R(\Pi, U)$



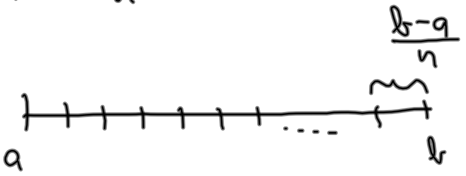
Maskevidden til partisjonen $a = x_0 < x_1 < \dots < x_n = b$

$$|\Pi| = \max\{x_i - x_{i-1} : i = 1, \dots, n\}$$

Setning: Anta at $f : [a,b] \to \mathbb{R}$ er integrerbar og at $\{\Pi_n, U_n\}$ er en følge av partisjoner og utplukk der $|\Pi_n| \to 0$. Da er

$$\lim_{n\to\infty} R(\Pi_n, U_n) = \int_a^b f(x)\,dx$$

Vanligste tilfellet: Π_n deler opp $[a,b]$ i n like store intervaller.

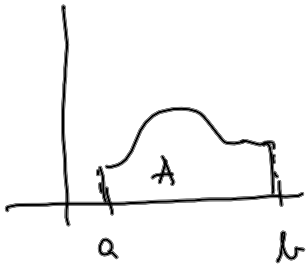


Slagordet: $\underbrace{\sum f(c_i)(x_i - x_{i-1})}_{\text{Riemann-sum.}} \xrightarrow[\text{oppdelingen blir finere.}]{\text{når}} \int f(x)\,dx$

Anvendelser av integralet

Arealberegninger:

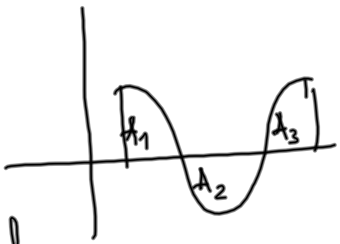
$$A = \int_a^b f(x)\,dx$$



$f(x) \geq 0$ når $x \in [a,b]$

Fare: Areal med fortegn:

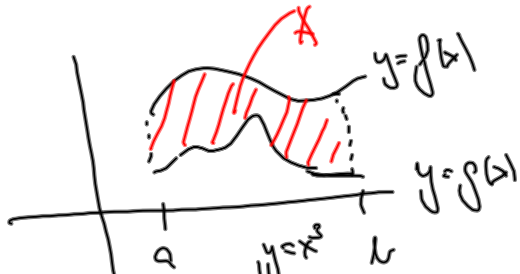
$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$



Areal mellom funksjonsgrafer

$f(x) \geq g(x)$ for alle $x \in [a,b]$

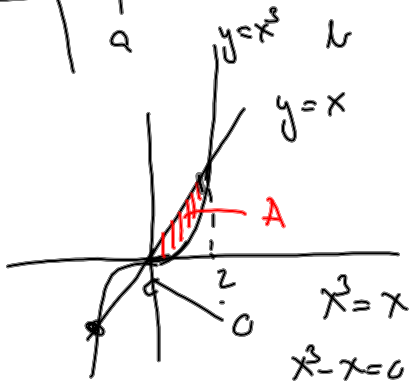
$$A = \int_a^b \left(f(x) - g(x)\right)dx$$



Eksempel: $f(x) = x,\ g(x) = x^3$

$$A = \int_0^1 (x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{4} = \underline{\underline{\frac{1}{4}}}$$



$x^3 = x$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

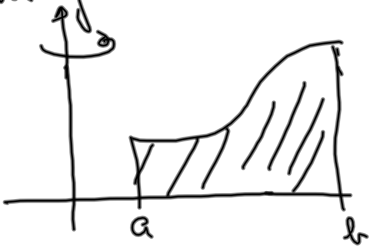
$x(x-1)(x+1) = 0$

$x = 0,\ x = 1,\ x = -1$

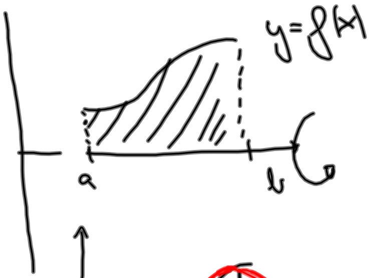
Volumer

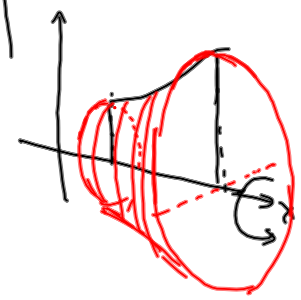
Omdreiningslegemer:

Om y-aksen

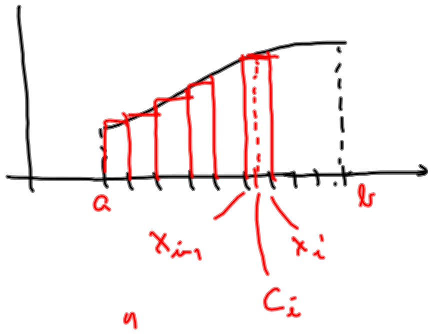


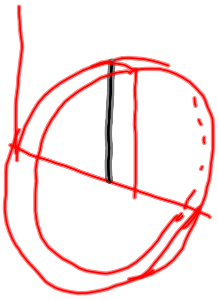
Om x-aksen





Omdreiningslegeme om x-aksen





$\pi r^2 h$

$$\sum_{i=1}^{n} \underbrace{\pi f(c_i)^2}\,(x_i - x_{i-1}) \longrightarrow \int_a^b \pi f(x)^2\,dx$$

Riemannsum til funksjonen $\pi f(x)^2$

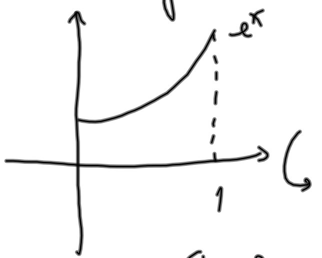
$$= \pi \int_a^b f(x)^2\,dx = V$$

Omdreiningslegeme om x-aksen:

$$V = \pi \int_a^b f(x)^2\,dx$$

Om x-aksen: $V = \pi \int_a^b f(x)^2\,dx$

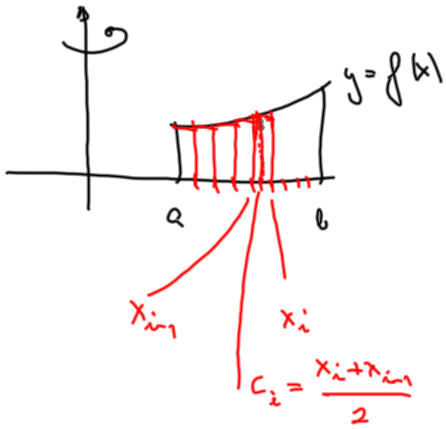
Eksempel: $f(x) = e^x \qquad 0 \le x \le 1$

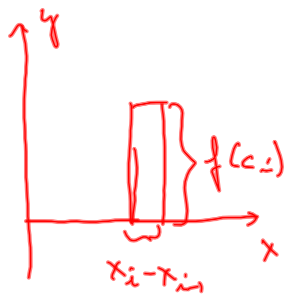


$$V = \pi \int_0^1 (e^x)^2\,dx = \pi \int_0^1 e^{2x}\,dx$$

$$= \pi \left[\frac{1}{2} e^{2x}\right]_0^1 = \frac{\pi}{2}\left[e^2 - e^0\right] = \underline{\frac{\pi}{2}(e^2 - 1)}$$

Omdreining om y-aksen



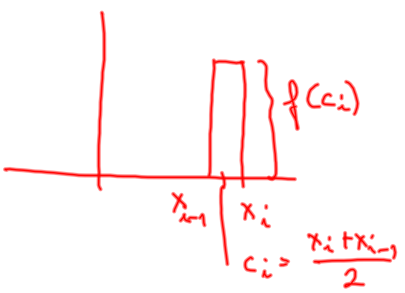


Volum av rør: $V_R = V_{\text{ytre syl}} - V_{\text{indre syl}}$

$$= \pi R^2 h - \pi r^2 h = \pi (R^2 - r^2) h$$

$$= \pi (R+r)(R-r) h = 2\pi \underbrace{\frac{R+r}{2}}_{\text{midtre radius}} (R-r) h$$

Våre rør:



$$V_i = 2\pi c_i (x_i - x_{i-1}) f(c_i)$$

Totalt volum

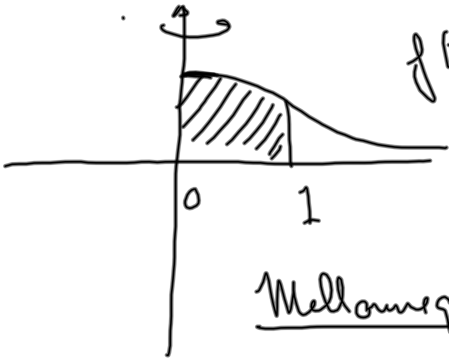
$$V \approx \sum 2\pi c_i (x_i - x_{i-1}) f(c_i)$$

$$= \sum \underbrace{2\pi c_i f(c_i)}_{2\pi x f(x)} (x_i - x_{i-1}) \to \int_a^b 2\pi x f(x)\,dx = 2\pi \int_a^b x f(x)\,dx$$

Omdreiningslegeme om y-aksen

$$V = 2\pi \int_a^b x f(x)\,dx$$

Eksempel:



$f(x) = \frac{1}{1+x^2}$, $0 \le x \le 1$, dreies om y-aksen.

$$V = 2\pi \int_0^1 x\, f(x)\, dx = 2\pi \int_0^1 \frac{x}{1+x^2}\, dx$$

Mellomregning: $\int \frac{x}{1+x^2}\, dx$

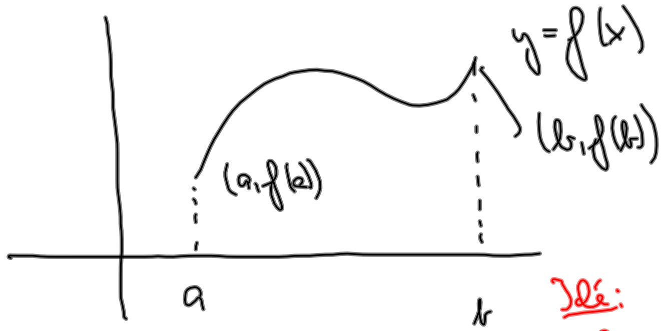
$u = 1 + x^2$

$$\underbrace{du = 2x\, dx}_{x\, dx = \frac{1}{2} du}$$

$$= \int \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln(1+x^2) + C$$

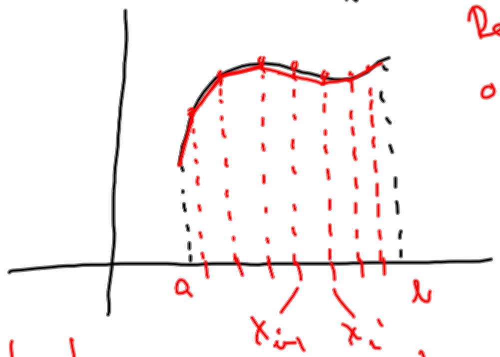
$$V = 2\pi \int_0^1 \frac{x}{1+x^2}\, dx = 2\pi \left[\frac{1}{2} \ln(1+x^2) \right]_0^1 = 2\pi \left(\frac{1}{2} \ln 2 - \frac{1}{2} \underbrace{\ln 1}_{0} \right) = \underline{\underline{\pi \ln 2}}$$

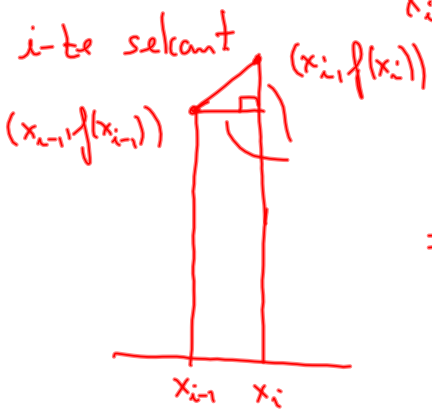
Buelengder



Hvor langt er det fra $(a, f(a))$ til $(b, f(b))$ langs funksjonsgrafen?

Idé: Regn ut lengden av sekantene og legg sammen.





Lengden til i-te sekant:

$$l_i = \sqrt{\underbrace{(x_i - x_{i-1})^2} + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\;\underbrace{(x_i - x_{i-1})}$$

$$\xrightarrow{\text{middelverdisetning}}$$

$$= \sqrt{1 + f'(c_i)^2}\,(x_i - x_{i-1})$$

Total lengde

$$L \approx \sum l_i = \sum_{i=1}^{n} \underbrace{\sqrt{1 + f'(c_i)^2}}\,(x_i - x_{i-1}) \longrightarrow \int_a^b \underline{\sqrt{1 + f'(x)^2}\,dx}$$

Riemannsum til $\sqrt{1 + f'(x)^2}$

<u>Formelen for buelengde:</u> Lengden fra $x = a$ til $x = b$ langs grafen $y = f(x)$ er

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

<u>Eksempel:</u> Finn lengden av kurven

$$f(x) = \ln(\cos x) \text{ fra } x=0 \text{ til } x=\frac{\pi}{4}.$$

$$f'(x) = \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x}$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1+f'(x)^2}\,dx$$

$$= \int_0^{\pi/4} \sqrt{1+\frac{\sin^2 x}{\cos^2 x}}\,dx = \int_0^{\pi/4} \sqrt{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}\,dx = \int_0^{\pi/4} \sqrt{\frac{1}{\cos^2 x}}\,dx$$

$$= \int_0^{\pi/4} \frac{1}{\cos x}\,dx = \int_0^{\pi/4} \frac{\cos x}{\cos^2 x}\,dx = \underline{\int_0^{\pi/4} \frac{\cos x}{1-\sin^2 x}\,dx}$$

$$u = \sin x$$
$$du = \cos x\,dx$$

<u>Mellomregning:</u> $\int \frac{\cos x}{1-\sin^2 x}\,dx = \int \frac{du}{\underline{1-u^2}} = \int \frac{du}{(1-u)(1+u)}$

$$= \int \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du = -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C$$

$$= -\frac{1}{2}\ln|1-\sin x| + \frac{1}{2}\ln|1+\sin x| + C$$

Tilbake:

$$L = \int_0^{\pi/4} \frac{\cos x}{1-\sin^2 x}\,dx = \Big[\qquad \Big]_0^{\pi/4}$$