Komplekse tall: $z = a + ib \,,\ a, b \in \mathbb{R} \,,\ i^2 = -1$

Eksempel: $\dfrac{2i}{i+z} = \dfrac{3}{2+z} \quad \Big| \cdot (i+z)(2+z)$

$$2i(2+z) = 3(i+z)$$

$$4i + 2iz = 3i + 3z \Rightarrow -3z + 2iz = -i \Rightarrow (-3+2i)z = -i$$

$$\Rightarrow \quad z = \frac{-i}{-3+2i} = \frac{-i\,(-3-2i)}{(-3+2i)(-3-2i)} = \frac{3i \overbrace{+2i^2}^{-2}}{\underbrace{(-3)^2 - (2i)^2}_{9+\ 4}} = \frac{-2+3i}{13}$$

$$= -\frac{2}{13} + i\,\frac{3}{13}$$

Konjugasjon: $z = a + ib$

Den konjugerte $\bar{z} = a - ib$

Eksempel: $z = 3 - 4i$

$\bar{z} = 3 + 4i$

$\Big(z = 3 - 4i = 3 + (-4)i$

$\bar{z} = 3 - (-4)i = 3 + 4i \Big)$

$$z\bar{z} = (a+ib)(a-ib) = a^2 - (ib)^2 = a^2 + b^2$$

Regneregler for konjugasjon:

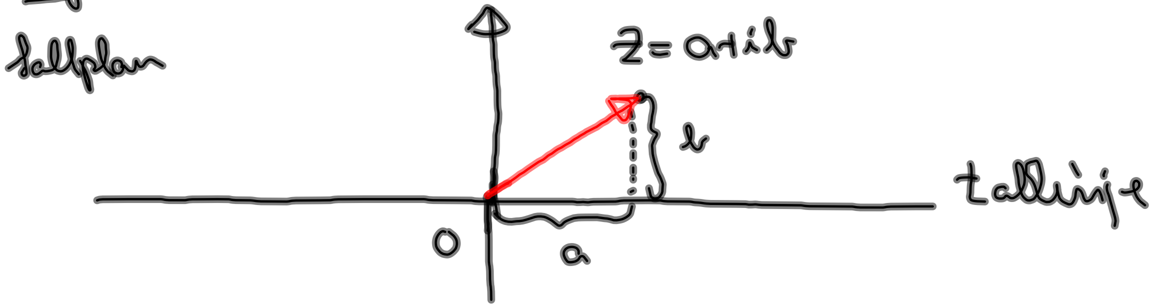
(i) $\overline{z + w} = \bar{z} + \bar{w}$

(ii) $\overline{z - w} = \bar{z} - \bar{w}$

(iii) $\overline{zw} = \bar{z}\,\bar{w}$

(iv) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\bar{z}}{\bar{w}}$

Geometrisk tolkning av komplekse tall (3.2)

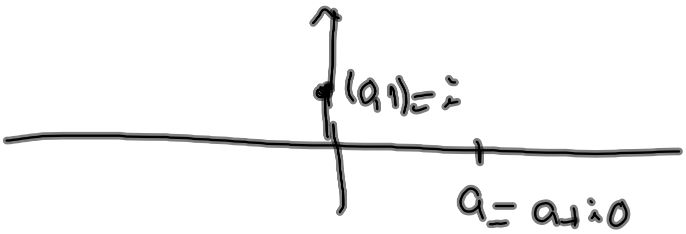


$z = a + ib$

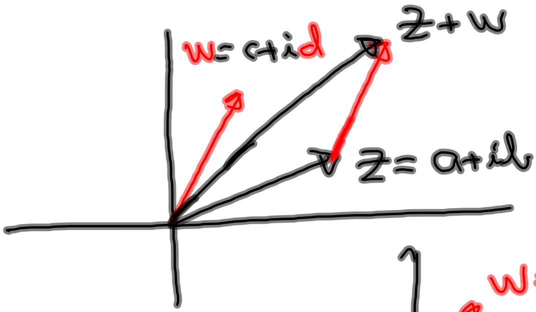
Det reelle tallet a: $z = a + i0$

$i = 0 + 1i$

$(0,1)$

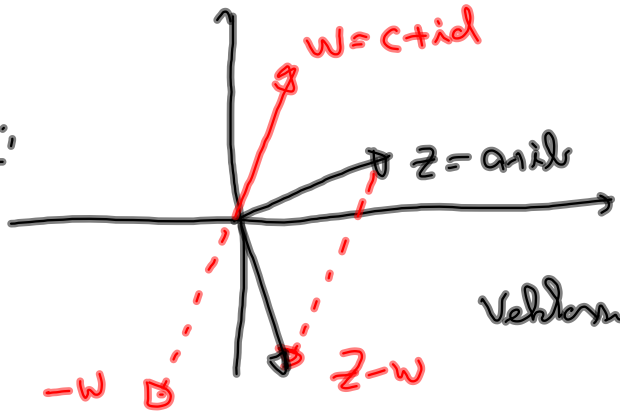


Addisjon: $z = a+ib$, $w = c+id$, $z+w = (a+c)+i(b+d)$



Vektoraddisjon

Subtraksjon:

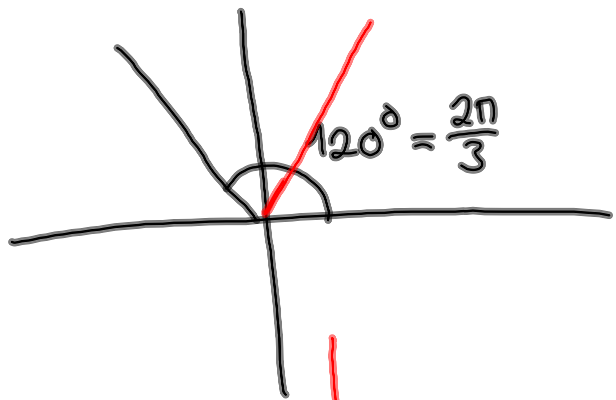


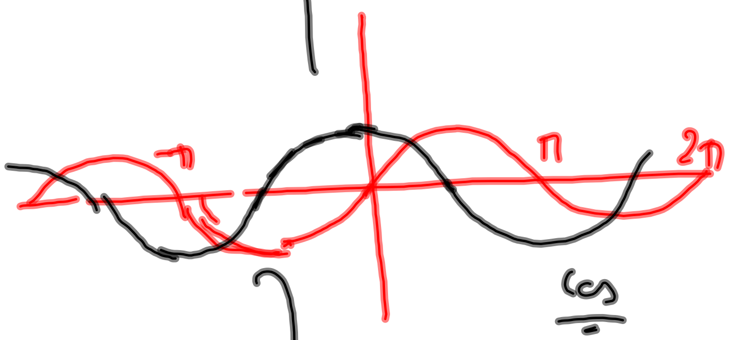
Vektorsubtraksjon

Hva med multiplikasjon og divisjon?

Digresjon: Trigonometri:

u	$\sin u$	$\cos u$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

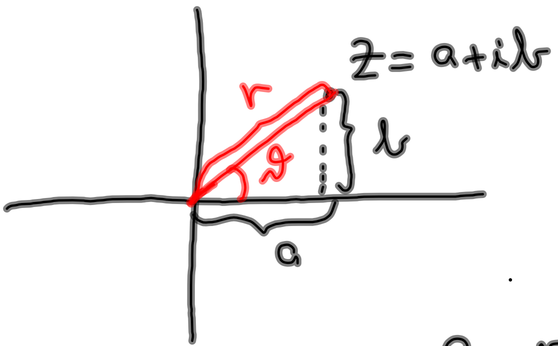




$$\left.\begin{aligned}
&\cos^2 u + \sin^2 u = 1\\
&\cos(u+v) = \cos u \cos v - \sin u \sin v\\
&\sin(u+v) = \sin u \cos v + \cos u \sin v
\end{aligned}\right\} \underline{\text{Husk}}$$

Polarform:

$\Theta = \vartheta$ (theta)



Polarkoordinater:

r: modulus til z

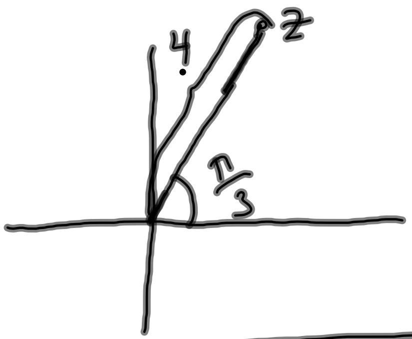
ϑ: argumentet til z

Sammenhenger: $a = r\cos\vartheta$, $b = r\sin\vartheta$

$r = \sqrt{a^2+b^2}$, $\sin\vartheta = \frac{b}{r}$

$\cos\vartheta = \frac{a}{r}$

Eksempel: $r = 4$, $\vartheta = \frac{\pi}{3}$ Hva er a og b?

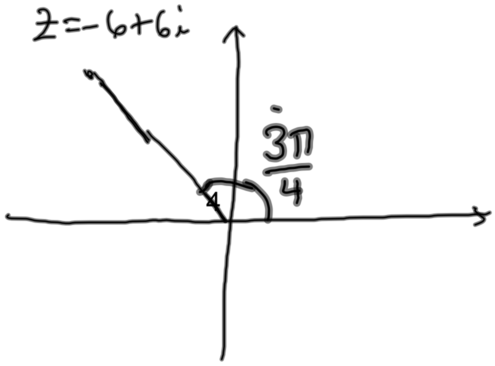


$a = r\cos\vartheta = 4\underbrace{\cos\frac{\pi}{3}}_{1/2} = 4\cdot\frac{1}{2} = 2$

$b = r\sin\vartheta = 4\underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}} = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}$

$$\boxed{z = 2 + 2i\sqrt{3}}$$

Eksempel: $z = -6 + 6i$, hva er r og θ?



$$r = \sqrt{a^2+b^2} = \sqrt{(-6)^2+6^2} = \sqrt{2 \cdot 36}$$

$$= 6\sqrt{2}$$

$$\left(\sqrt{2} \cdot \sqrt{36} = \sqrt{2} \cdot 6 \right)$$

$$\sin\theta = \frac{b}{r} = \frac{6}{6\sqrt{2}}$$

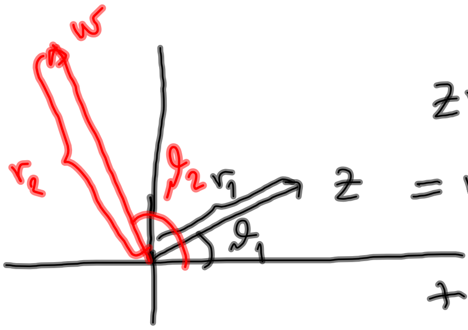
$$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Hvilken vinkel i annen kvadrant har sinus lik $\frac{\sqrt{2}}{2}$? $\theta = \frac{3\pi}{4}$

Kompleks multiplikasjon:

$$z = a + ib = r_1 \cos\theta_1 + i\, r_1 \sin\theta_1$$

$$w = c + id = r_2 \cos\theta_2 + i\, r_2 \sin\theta_2$$



$$zw = (r_1 \cos\theta_1 + i\, r_1 \sin\theta_1)(r_2 \cos\theta_2 + i\, r_2 \sin\theta_2)$$

$$= r_1 r_2 \cos\theta_1 \cos\theta_2 + i\, r_1 r_2 \cos\theta_1 \sin\theta_2$$

$$+ i\, r_1 r_2 \sin\theta_1 \cos\theta_2 - r_1 r_2 \sin\theta_1 \sin\theta_2$$

$$= r_1 r_2 \underbrace{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)}_{\cos(\theta_1+\theta_2)} + i\, r_1 r_2 \underbrace{(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)}_{\sin(\theta_1+\theta_2)}$$

modulus

argument

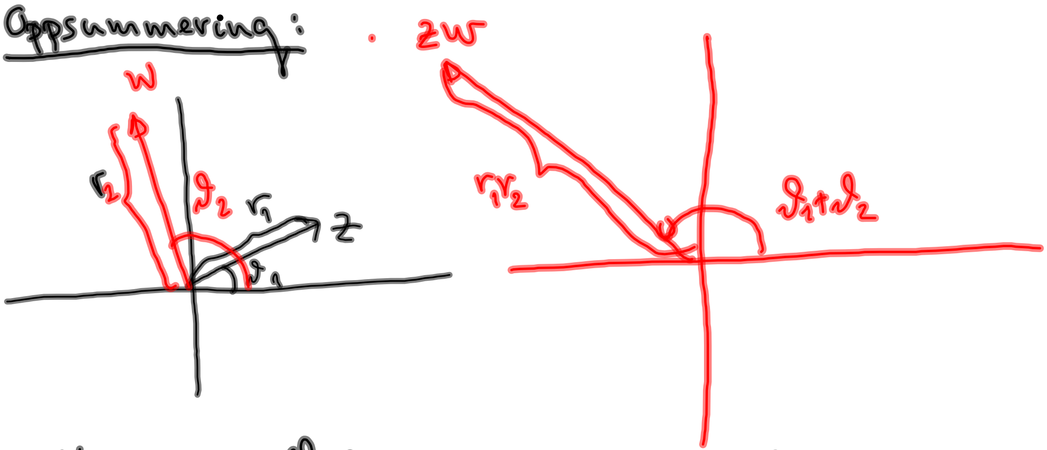
$$= r_1 r_2 \cos(\theta_1+\theta_2) + i\, r_1 r_2 \sin(\theta_1+\theta_2)$$

Polarform

$z = r\cos\theta + i\, r \sin\theta$

zw er altså et komplekst tall med modulus $r_1 r_2$ og argument $\theta_1 + \theta_2$!

Oppsummering:

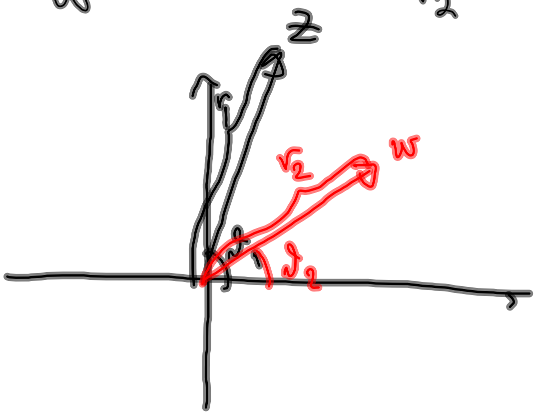


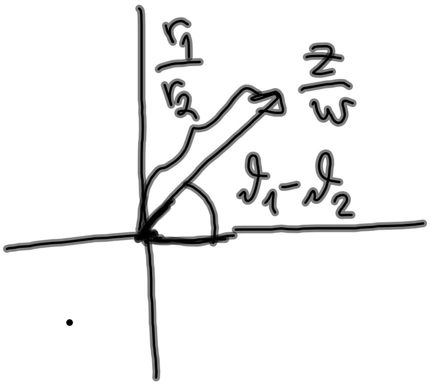
Når vi multipliserer to komplekse tall, adderer vi argumentene og multipliserer modulusene!

Division: z med modulus r_1 argument ϑ_1

w —"— r_2 —"— ϑ_2

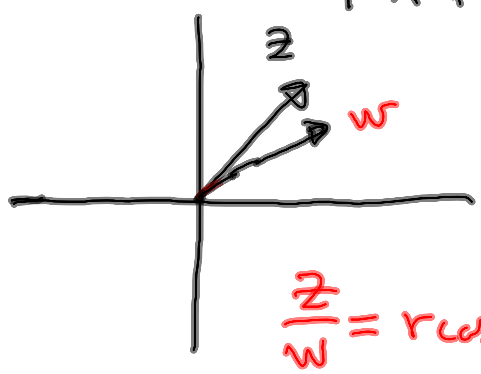
$\frac{z}{w}$: modulus $\frac{r_1}{r_2}$, argument $\vartheta_1 - \vartheta_2$





Eksempel: $z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $w = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$r_1 = 1,\ \vartheta_1 = \frac{\pi}{4}$ $\qquad$ $r_2 = 1,\ \vartheta_2 = \frac{\pi}{6}$



$\frac{z}{w}$: modulus $\frac{r_1}{r_2} = \frac{1}{1}$

argument $\vartheta_1 - \vartheta_2 = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12} = 15^\circ$

$$\frac{z}{w} = r\cos\vartheta + ir\sin\vartheta = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$$

Alternativt:

$$\frac{z}{w} = \frac{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2} + i\frac{1}{2}} = \frac{(\sqrt{2} + i\sqrt{2})(\sqrt{3} - i)}{(\sqrt{3} + i)(\sqrt{3} - i)} =$$

$$= \frac{\sqrt{6} - i\sqrt{2} + i\sqrt{6} + \sqrt{2}}{3+1} = \underbrace{\frac{\sqrt{6}+\sqrt{2}}{4}}_{\cos\frac{\pi}{12} =} + i\underbrace{\frac{\sqrt{6}-\sqrt{2}}{4}}_{= \sin\frac{\pi}{12}}$$