Denne uken: Plenumsregning onsdag
forelesning torsdag

Skifte av variabel

$$\int f(g(x))\,dx = \int f(u)h'(u)\,du \Big|_{u=g(x)}$$

der $h(u)$ er den omvendte funksjonen av $g(x)$

$$\boxed{\begin{aligned} &u = g(x) \Rightarrow x = h(u) \\ &dx = h'(u)\,du \end{aligned}}$$

$$\int_a^b f(g(x))\,dx = \int_{g(a)}^{g(b)} f(u)h'(u)\,du$$

Eksempel. $I = \int_{1/2}^{\sqrt{3}/2} \frac{\sqrt{1-x^2}}{x^2}\,dx$

$x = \sin u$, $u = \arcsin x$

$dx = \cos u\,du$

$x = 1/2$, $u = \arcsin 1/2 = \frac{\pi}{6}$

$x = \frac{\sqrt{3}}{2}$, $u = \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\overset{\cos u}{\sqrt{1-\sin^2 u}}}{\sin^2 u} \cos u\,du$$

$$= \int_{\frac{\pi}{6}}^{\pi/3} \frac{\cos^2 u}{\sin^2 u}\,du = \int_{\frac{\pi}{6}}^{\pi/3} \frac{1-\sin^2 u}{\sin^2 u}\,du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{\sin^2 u} - 1\right) du = \Big[-\cot u - u\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\cot\frac{\pi}{3} - \frac{\pi}{3} + \cot\frac{\pi}{6} + \frac{\pi}{6}$$

$$= -\frac{\sqrt{3}}{3} - \frac{\pi}{3} + \underset{\frac{3\sqrt{3}}{3}}{\underset{\shortparallel}{\sqrt{3}}} + \frac{\pi}{6} = \underline{\underline{-\frac{\pi}{6} + \frac{2\sqrt{3}}{3}}}$$

$$\boxed{\begin{aligned} &\cot\frac{\pi}{3} = \frac{1}{\tan\frac{\pi}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ &\cot\frac{\pi}{6} = \frac{1}{\tan\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \underline{\sqrt{3}} \end{aligned}}$$

Dellbrøksoppspalting

Integrasjon av rasjonale funksjoner: $R(x) = \frac{P(x)}{Q(x)}$ > polynomer

Enkleste variant: $\int \frac{Ax+B}{(x-a)(x-b)}\,dx$

Idé: $\frac{Ax+B}{(x-a)(x-b)} = \underbrace{\frac{C}{x-a} + \frac{D}{x-b}}$ dellbrøk oppspalting: Finn C og D slik at dette stemmer.

Eksempel: $\int \frac{x+8}{x^2+x-6}\,dx$

Dellbrøk oppspalting:

$$\cancel{\frac{x+8}{x^2+x-6}} = \frac{x+8}{(x-2)(x+3)} = \frac{C}{x-2} + \frac{D}{x+3}$$

Faktoriser x^2+x-6:

$$x = \frac{-1 \pm \sqrt{1^2-4\cdot 1(-6)}}{2\cdot 1} = \frac{-1\pm 5}{2} = \begin{cases} 2 \\ -3 \end{cases}$$

$$x^2+x-6 = (x-2)(x+3)$$

Gange med $(x-2)(x+3)$:

$$x+8 = C(x+3) + D(x-2)$$
$$= (C+D)x + 3C - 2D$$

Krever: $C+D = 1$, $3C-2D = 8$

$$3C - 2D = 8 \quad (\mathrm{I})$$
$$2C + 2D = 2 \quad (2\cdot\mathrm{I})$$

$$5C = 10 \Rightarrow C = 2$$

$$C+D = 1 \Rightarrow D = -1$$

Dermed har vi

$$\frac{x+8}{(x-2)(x+3)} = \frac{2}{x-2} - \frac{1}{x+3}$$

Altså:

$$\int \frac{x+8}{x^2+x-6}\,dx = \int \left(\frac{2}{x-2} - \frac{1}{x+3}\right)dx = 2\int \frac{1}{\underbrace{x-2}_{u}}\,dx - \int \frac{1}{\underbrace{x+3}_{v}}\,dx$$

$$= 2\ln|x-2| - \ln|x+3| + C$$

Delbrøkoppspalting generelt: $\int \frac{P(x)}{Q(x)}\,dx$ P, Q polynomer

Faktoriseres:

$$Q(x) = \underbrace{(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots\cdots}_{\text{førstegradsfaktorer}}\ \underbrace{(x^2+a_1x+b_1)^{n_1}(x^2+a_2x+b_2)^{n_2}\cdots\cdots}_{\text{annengradsfaktorer.}}$$

$$\frac{P(x)}{(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x^2+a_1x+b_1)^{n_1}(x^2+a_2x+b_2)^{n_2}\cdots} =$$

$$= \frac{A_1}{x-r_1} + \frac{A_2}{(x-r_1)^2} + \cdots + \frac{A_{m_1}}{(x-r_1)^{m_1}} + \quad \Big\}\ \text{ledd tilhørende faktoren } (x-r_1)^{m_1}$$

$$+ \frac{B_1}{x-r_2} + \frac{B_2}{(x-r_2)^2} + \cdots + \frac{B_{m_2}}{(x-r_2)^{m_2}} + \quad \Big\}\ \text{ledd tilhørende } (x-r_2)^{m_2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{C_1x+D_1}{x^2+a_1x+b_1} + \frac{C_2x+D_2}{(x^2+a_1x+b_1)^2} + \cdots + \frac{C_{n_1}x+D_{n_1}}{(x^2+a_1x+b_1)^{n_1}} + \quad \Big\}\ \text{ledd tilhørende } (x^2+a_1x+b_1)^{n_1}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

Eksempel: $$\frac{3x^3-7x^2+2}{(x-1)(x+4)^3(x^2+4x+9)^2}$$

$$= \frac{A_1}{(x-1)} + \frac{B_1}{(x+4)} + \frac{B_2}{(x+4)^2} + \frac{B_3}{(x+4)^3} + \frac{C_1x+D_1}{x^2+4x+9} + \frac{C_2x+D_2}{(x^2+4x+9)^2}$$

Må forsøke å finne konstanter slik at likheten holder for alle "meningsfulle" x.

Fremgangsmåte: Gang med $(x-1)(x+4)^3(x^2+4x+9)^2$ på begge sider, og sett koeffisientene lik hverandre. Dette gir ligningssystemer for å finne A_1, B_1, B_2 osv.

Eksempel: $$\frac{5x^3-10x^2+8x-2}{(x-1)^2(x^2-x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2-x+1}$$

Ganger med fellesnevner:

$$5x^3-10x^2+8x-2 = A(x-1)(x^2-x+1) + B(x^2-x+1) + (Cx+D)\underbrace{(x-1)^2}_{(x^2-2x+1)}$$

$$= A x^3 - Ax^2 + Ax - Ax^2 + Ax - A + Bx^2 - Bx + B$$

$$+ Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

$$= (A+C)x^3 + (-2A+B-2C+D)x^2 + (2A-B+C-2D)x + (-A+B+D)$$

Ligninger: $A+C=5$ (I), $-2A+B-2C+D=-10$ (II); $2A-B+C-2D=8$ (III); $-A+B+D=-2$ (IV)

$C=5-A$; Setter inn i ligning II:

$$-2A+B-2(5-A)+D=-10$$

$$-\cancel{2A}+B-\cancel{10}+\cancel{2A}+D=-\cancel{10}$$

$$B+D=0$$

Setter inn i ligning III:

$$2A-B+5-A-2D=8$$

$$A-B-2D=3$$

Setter igjen med:

$B+D=0$

$\left.\begin{matrix} A-B-2D=3 \\ -A+B+D=-2 \end{matrix}\right\} \Rightarrow$ Legger sammen: $-D=1 \Rightarrow D=-1$

Fra øverste ligning $B=1$.

$A=3+B+2D=3+1-2=\underline{2}$

$C=5-A=5-2=\underline{\underline{3}}$

Oppsummerer:

$$\frac{5x^3-10x^2+8x-2}{(x-1)^2(x^2-x+1)} = \cancel{\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2-x+1}} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \underline{\frac{3x-1}{x^2-x+1}}$$

Oppsummering: Anta at $P(x), Q(x)$ er polynomer der graden til $P(x)$ er mindre enn graden til $Q(x)$. Hvis $Q(x) = (x-r_1)^{m_1} \cdots\cdots (x^2+a_1x+b_1)^{n_1} \cdots$, så kan vi alltid finne konstanter slik at

ledd fra andre førstegradsfaktorer

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-r_1)} + \cdots + \frac{A_{m_1}}{(x-r_1)^{m_1}} + \underbrace{\cdots\cdots}_{} + \frac{C_1x+D_1}{x^2+a_1x+c_1} + \cdots + \frac{C_{n_1}x+D_{n_1}}{(x^2+a_1x+b_1)^{n_1}}$$

ledd fra andre annengradsfaktorer

$$+ \underbrace{\cdots\cdots}_{}$$

For å kunne videre med integrasjonen, må vi kunne integrere uttrykk av typen:

$$\int \frac{A_1}{x-r_1}\,dx = A_1 \ln|x-r_1| + C$$

$$\int \frac{A_k}{(x-r_1)^k}\,dx = \int A_k (x-r_1)^{-k}\,dx = \frac{A_k}{-k+1}(x-r_1)^{-k+1} + C$$

$$\int \frac{C_1x+D_1}{x^2+a_1x+b_1}\,dx = \text{ skal gjøre med neste gang!} \longleftarrow$$

$$\int \frac{C_1x+D}{(x^2+a_1x+b_1)^{n_1}}\,dx = \text{ Stykkelig grisevis}$$