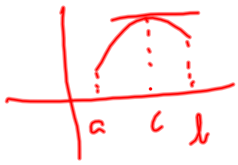
Maks/min-problemer

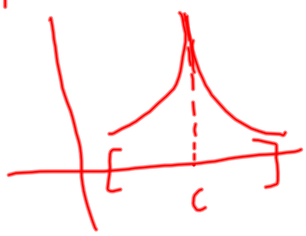
Dersom en kontinuerlig funksjon $f:[a,b]\to\mathbb{R}$ har et maks- eller min-punkt i c, så må enten

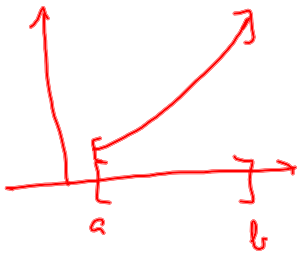
(i) c være et punkt der $f'(c)=0$

(ii) c være et punkt der $f'(c)$ ikke finnes

(iii) c er et endepunkt i intervallet

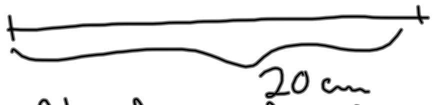




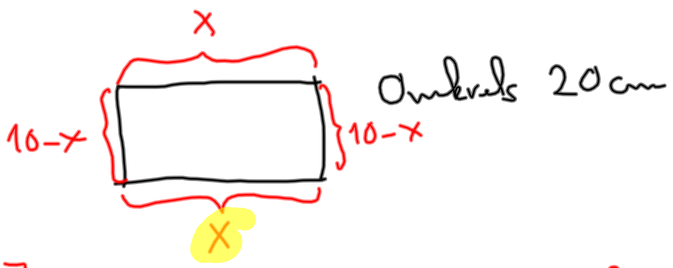


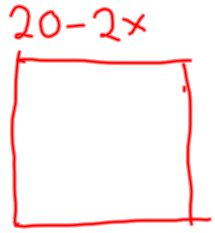
Uoppstilte maks- og min-problemer

Eksempel: Ståltråd



Bøy til et rektangel slik at arealet blir så stort som mulig.



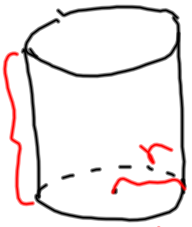


$x \in [0,10]$ $\quad A(x) = x(10-x) = 10x - x^2$

$A'(x) = 10 - 2x$, $\quad A'(x) = 0$: $\quad 10 - 2x = 0$, $\quad \underline{\underline{x = 5}}$

Størst areal når $x = 5$ og rektangelet er et kvadrat.

Eksempel:



~~Innhold~~ Volum: 1 dm³

Krum flate dobbelt så dyr per kvadratcentimeter som topp og bunn.

Finn boksen som minimerer materialkostnadene.

Pris per kvadratdesimeter: p topp/bunn
2p sidekant.

$$1 = V = \pi r^2 h \Rightarrow h = \frac{1}{\pi r^2}$$

Kostnaden bunn + topp

$$K = 2p\pi r^2 + 2p\,2\pi r h$$

$$= 2p\pi r^2 + 4p\pi r h = 2p\pi r^2 + 4p\cancel{\pi}\cancel{r}\frac{1}{\cancel{\pi} r^{\cancel{2}}}$$

$$= 2p\pi r^2 + \frac{4p}{r}$$

Deriver: $K'(r) = 2p\pi 2r - \frac{4p}{r^2} = 4p\pi r - \frac{4p}{r^2}$

$$0 = K'(r) = 4p\pi r - \frac{4p}{r^2} \Rightarrow \pi r - \frac{1}{r^2} = 0 \Rightarrow r^3 = \frac{1}{\pi} \Rightarrow r = \sqrt[3]{\frac{1}{\pi}}$$

$$= \frac{1}{\sqrt[3]{\pi}}.$$

<u>Eksempel:</u> En bil har forbruket

$$2 + 0.08v^2 \text{ liter/time når farten er } v.$$

Hva fart må man kjøre for å ha minst forbruk per kilometer?

Hva lang tid bruker vi på en kilometer? $s = vt$, $t = \frac{s}{v}$

$$t = \frac{1}{v}$$

<u>Bensinforbruk på 1 km:</u> $\underbrace{(2+0.08v^2)}_{\text{forbruk/time}}\underbrace{\frac{1}{v}}_{\text{antall timer}} = \frac{2}{v} + 0.08v$

$$F(v) = \frac{2}{v} + 0.08v$$

$$F'(v) = -\frac{2}{v^2} + 0.08 \qquad F'(v)=0: \; -\frac{2}{v^2} + 0.08 = 0 \Rightarrow$$

$$0.08\, v^2 = 2 \Rightarrow v^2 = \frac{2}{0.08} = \frac{200}{8} = \underline{25}$$

$$\underline{v = 5 \text{ km/t}}$$

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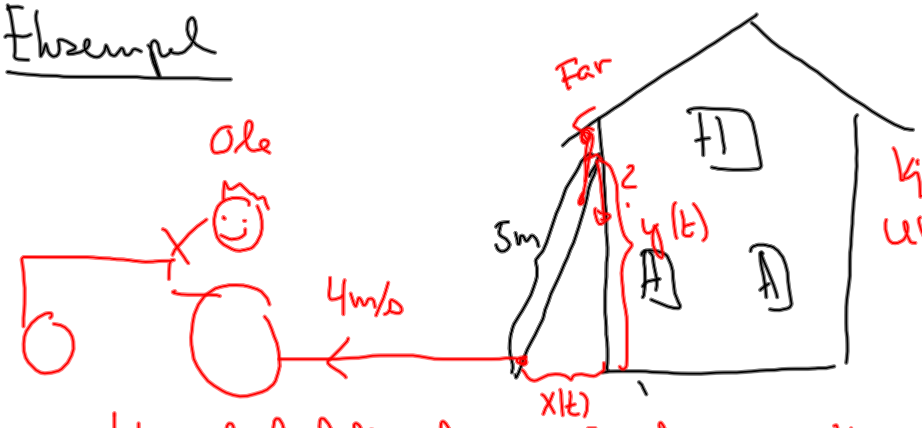
Koblede hastigheter

Én kjent, én ukjent hastighet: Finn den ukjente!

Eksempel



Kjenner $x'(t)$

Ukjent $y'(t)$

Hvor fort faller far når han er $\underline{4\text{m}}$ over bakken?

Generell sammenheng mellom $x(t)$ og $y(t)$:

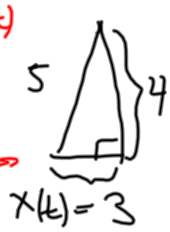
$$x(t)^2 + y(t)^2 = 25$$

Deriverer:

$$\cancel{2}x(t)x'(t) + \cancel{2}y(t)y'(t) = 0$$

Løser for $y'(t)$:

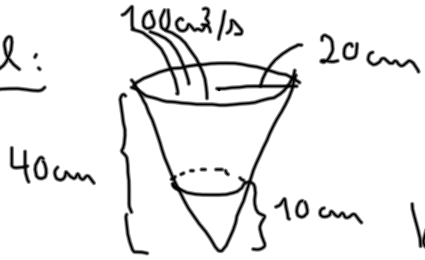
$$y'(t) = -\frac{x(t)}{\underbrace{y(t)}_{4\text{m}}}\, x'(t) \quad \leftarrow 4\,\text{m/s}$$



Bruker Pythagoras til å vise at $x(t)=3$ når $y(t)=4$:

$$y'(t) = -\frac{3}{4}\,4 = \underline{\underline{-3\,\text{m/s}}}$$

Eksempel:



Hva fort stiger høyden i det øyeblikket den er 10 cm?

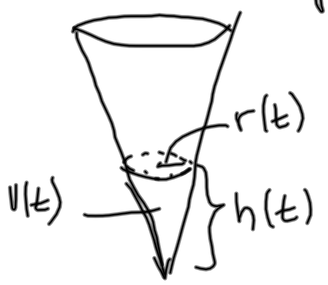
Kjent hastighet: Volumøkning/tidsenhet

Ukjent hastighet: høydeøkning

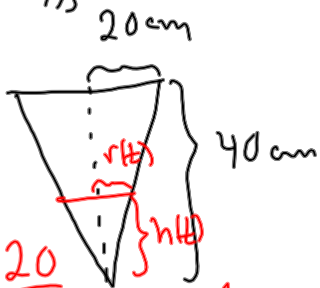
$$V'(t) = 100 \text{ cm}^3/\text{s}$$

$$h'(t) = ?$$

Søker en sammenheng mellom V(t) og h(t).



$$V(t) = \frac{1}{3}\pi r(t)^2 h(t)$$



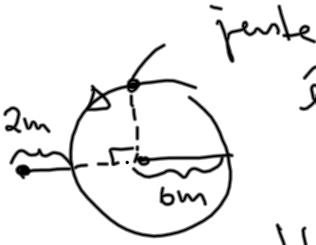
$$\frac{r(t)}{h(t)} = \frac{20}{40} \Rightarrow r(t) = \frac{1}{2}h(t)$$

$$V(t) = \frac{1}{3}\pi\left(\frac{h(t)}{2}\right)^2 h(t) = \frac{1}{12}\pi h(t)^3$$

Deriverer: $$V'(t) = \frac{\pi}{12} 3h(t)^2 h'(t) = \frac{\pi}{4} h(t)^2 h'(t)$$

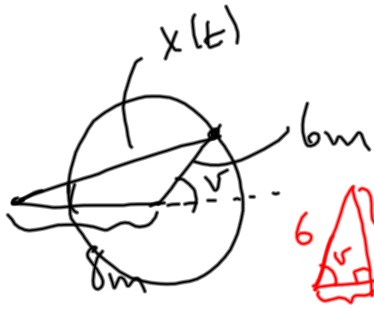
dvs: $$h'(t) = \frac{V'(t)}{\frac{\pi}{4}h(t)^2} = \frac{4V'(t)}{\pi h(t)^2} = \frac{4\cdot 100}{\pi \cdot 10^2} = \frac{4}{\pi}\text{ cm/s.}$$

Eksempel:



Én omdreining i minuttet

Generell bilde



$\frac{h}{6} = \sin v$

$\frac{g}{6} = \cos v$

Hva fort endrer avstanden mellom far og datter seg?

Pythagoras:

$$x(t)^2 = (8+6\cos v)^2 + (6\sin v)^2$$

$$= 64 + 96\cos v + 36\cos^2 v + 36\sin^2 v$$

$$= 64 + 96\cos v(t) + 36$$

$$= 100 + 96\cos v(t)$$

Deriver: $2x(t)x'(t) = -96\sin v(t) \cdot v'(t)$

$$x'(t) = -\frac{48\overbrace{\sin v(t)}^{=1}}{\underbrace{x(t)}_{10}}\underbrace{v'(t)}_{\to 2\pi}$$

Interessant i punktet $v(t) = \frac{\pi}{2}$

$$x(t)^2 = 100 + 96\underbrace{\cos\frac{\pi}{2}}_{0} = 100$$

$$x'(t) = \frac{-48}{10}2\pi = -\frac{96}{10}\pi = -\underline{\underline{\frac{48\pi}{5}}}\ \text{m/min}$$