Delbrøkoppspalting

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{P(x)}{\underbrace{(x-a_1)^{n_1}\cdots\cdots}_{\text{1. gradsfaktorer}}\underbrace{(x^2+b_1x+c_1)^{m_1}\cdots\cdots}_{\text{2 gradsfaktorer}}}\,dx$$

$$= \int \frac{A_1}{(x-a_1)}\,dx + \int \frac{A_2}{(x-a_1)^2}\,dx + \cdots + \int \frac{A_{n_1}}{(x-a_1)^{n_1}}\,dx \quad \Big\}\ \text{grei å integrere}$$

Skal lære oss! Grufe greier!

$$+ \cdots$$

$$+ \int \frac{B_1x+C_1}{(x^2+b_1x+c_1)}\,dx + \int \frac{B_2x+C_2}{(x^2+b_1x+c_1)^2} + \cdots + \int \frac{B_{m_1}x+C_{m_1}}{(x^2+bx+c)^{m_1}}\,dx$$

$$+ \cdots$$

Må vite hvordan man integrerer uttrykk av typen

$$\frac{Bx+C}{x^2+bx+c}$$ der x^2+bx+c ikke kan faktoriseres reelt.

Eksempel: $\displaystyle\int \frac{3}{\underbrace{x^2+4x}_{\text{gjøre fullstendig}}+8}\,dx =$

$$= \int \frac{3}{\underbrace{x^2+4x+4}_{(x+2)^2}\underbrace{-4+8}_{4}}\,dx = \int \frac{3}{(x+2)^2+4}\,dx$$

$$= \int \frac{3}{4\left(\frac{(x+2)^2}{4}+1\right)}\,dx = \frac{3}{4}\int \frac{1}{\left(\left(\frac{x+2}{2}\right)^2+1\right)}\,dx$$

$$= \frac{3}{\cancel{4}\,2}\int \frac{\cancel{2}\,du}{u^2+1} = \frac{3}{2}\arctan u + C$$

$u = \frac{x+2}{2}$

$du = \frac{1}{2}\,dx$

$dx = 2\,du$

$$= \frac{3}{2}\arctan\frac{x+2}{2} + C$$

Generell:

x^2+ax+b

$= \underbrace{x^2+ax+\left(\frac{a}{2}\right)^2}-\left(\frac{a}{2}\right)^2+b$

$(x+\frac{a}{2})^2+b-\left(\frac{a}{2}\right)^2$

Eksempel: $\int \frac{4x+1}{x^2+6x+18}\,dx =$

$u = x^2+6x+18$

$u' = 2x+6$, $du = (2x+6)\,dx$

$= 2\int \frac{2x+\frac{1}{2}}{x^2+6x+18}\,dx = 2\int \frac{2x+6-6+\frac{1}{2}}{x^2+6x+18}\,dx$ $\quad \frac{11}{2}$

$= 2\int \frac{2x+6}{\underbrace{x^2+6x+18}_{u}}\,dx - \int \frac{11}{x^2+6x+18}\,dx$ $\quad du$ — Hurra, dette kan vi løse!

$= 2\int \frac{du}{u} - \int \frac{11}{x^2+6x+18}\,dx = 2\ln|u| - \int \frac{11}{x^2+6x+18}\,dx$

$= 2\ln(x^2+6x+18) - \int \frac{11}{x^2+6x+18}\,dx$

Mellomregning: $\int \frac{11}{x^2+6x+18} = \int \frac{11}{\underbrace{x^2+6x+9}_{(x+3)^2}\underbrace{-9+18}_{9}} = \int \frac{11}{(x+3)^2+9}\,dx$

$= \frac{11}{9}\int \frac{dx}{\frac{(x+3)^2}{9}+1}\,dx = \frac{11}{9}\int \frac{dx}{\left(\frac{x+3}{3}\right)^2+1}$ $\qquad z = \frac{x+3}{3}$

$dz = \frac{1}{3}dx \Rightarrow dx = 3dz$

$= \frac{11}{\cancel{9}\,3}\int \frac{\cancel{3}\,dz}{z^2+1} = \frac{11}{3}\arctan z + C = \underline{\frac{11}{3}\arctan\frac{x+3}{3} + C}$

Hele integralet: $\int \frac{4x+1}{x^2+6x+18}\,dx = \underline{\underline{2\ln(x^2+6x+18) - \frac{11}{3}\arctan\frac{x+3}{3} + C}}$

Tilbake til gammelt eksempel: $I = \int \frac{5x^3-10x^2+8x-2}{(x-1)^2(x^2-x+1)}\,dx$

Fra forige:

$$\frac{5x^3-10x^2+8x-2}{(x-1)^2(x^2-x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2-x+1}$$

$$= \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{3x-1}{x^2-x+1}$$

$$\left(\frac{1}{u}\right)' = -\frac{1}{u^2}$$

Dermed:

$$I = \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx + \int \frac{3x-1}{x^2-x+1}\,dx$$

$$= 2\ln|x-1| - \frac{1}{(x-1)} + \int \frac{3x-1}{x^2-x+1}\,dx$$

Mellomregning: $I_2 = \int \frac{3x-1}{x^2-x+1}\,dx$

$u = x^2-x+1$

$u' = 2x-1$

$du = (2x-1)\,dx$

$$= \frac{3}{2}\int \frac{2x-\frac{2}{3}}{x^2-x+1}\,dx = \frac{3}{2}\int \frac{2x-1+1-\frac{2}{3}}{x^2-x+1}\,dx \qquad 1-\frac{2}{3} = \frac{1}{3}$$

$$= \frac{3}{2}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{\cancel{3}}{2}\int \frac{\frac{1}{\cancel{3}}}{x^2-x+1}\,dx$$

$$= \frac{3}{2}\int \frac{du}{u} + \frac{1}{2}\int \frac{1}{x^2-x+1}\,dx = \frac{3}{2}\ln|u| + \frac{1}{2}\int \frac{dx}{x^2-x+1}$$

$$= \frac{3}{2}\ln(x^2-x+1) + \frac{1}{2}\int \frac{dx}{x^2-x+1}$$

Mellomregning: $\frac{1}{2}\int \frac{dx}{x^2-x+1} = \frac{1}{2}\int \frac{dx}{\underbrace{x^2-x+\frac{1}{4}}_{(x-\frac{1}{2})^2}-\frac{1}{4}+1}$

$$= \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} = \frac{1}{2}\int \frac{dx}{\frac{3}{4}\left(\underbrace{\frac{4}{3}(x-\frac{1}{2})^2}_{\left(\frac{2x-1}{\sqrt{3}}\right)^2}+1\right)} = \frac{1}{\cancel{2}}\cdot\frac{\overset{2}{\cancel{4}}}{3}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}$$

!!please!

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$$= \frac{2}{3}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}$$

$v = \frac{2x-1}{\sqrt{3}}$

$v' = \frac{2}{\sqrt{3}}$, $dv = \frac{2}{\sqrt{3}}\,dx$

$dx = \frac{\sqrt{3}}{2}\,dv$

$$= \frac{\cancel{2}}{3}\int \frac{\frac{\sqrt{3}}{\cancel{2}}\,dv}{v^2+1} = \frac{\sqrt{3}}{3}\arctan v + C$$

$$= \frac{\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} + C$$

Oppsummering: $\int \frac{5x^3-10x^2+8x-2}{(x-1)^2(x^2-x+1)}\,dx = 2\ln|x-1| - \frac{1}{x-1} +$

$$+ \frac{3}{2}\ln(x^2-x+1) + \frac{\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} + C$$

Hva med delbrøkoppspalting når graden i teller ikke er mindre enn graden i nevner? $\frac{P(x)}{Q(x)}$

Polynomdivisjon

$$\frac{P(x)}{Q(x)} = \underbrace{p(x)}_{\text{polynom}} + \frac{R(x)}{Q(x)}$$

— rest med grad mindre enn Q.

Så

$$\int \frac{P(x)}{Q(x)}\,dx = \int p(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$$

delbrøkoppspalting etter vanlige regler.

Eksempel: $\int \frac{x^3+2x^2+3x+1}{x+3}\,dx$

Polynomdivisjon:

$$
\begin{aligned}
&x^3+2x^2+3x+1 : x+3 = x^2-x+6 \quad \text{— } p(x)\\
&-(x^3+3x^2)\\
&\quad -x^2+3x+1\\
&\quad -(-x^2-3x)\\
&\qquad 6x+1\\
&\qquad -(6x+18)\\
&\qquad\quad -17 \quad \text{— } R(x)
\end{aligned}
$$

Altså:

$$\frac{x^3+2x^2+3x+1}{x+3} = x^2-x+6 + \frac{-17}{x+3}$$

Dermed er

$$\int \frac{x^3+2x^2+3x+1}{x+3}\,dx = \int (x^2-x+6)\,dx - \int \frac{17}{x+3}\,dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 6x - 17\ln|x+3| + C$$

Oppskrift på delbrøksoppspalting

$$\int \frac{P(x)}{Q(x)}\,dx$$

1. Hvis ikke graden til telleren er mindre enn graden til nevneren, så polynomdivider

$$\frac{P(x)}{Q(x)} = p(x) + \frac{R(x)}{Q(x)}$$

og integrer opp $p(x)$ slik at vi sitter igjen med $\int \frac{R(x)}{Q(x)}dx$

2. Faktoriser $Q(x)$:

$$Q(x) = (x-a_1)^{n_1} \cdot \ldots \cdot (x^2+b_1x+c_1)^{m_1} \cdot \ldots$$

3. Sett opp delbrøksoppspaltingen:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + \ldots + \frac{A_{n_1}}{(x-a_1)^{n_1}} +$$

$$+ \ldots$$

$$+ \frac{B_1x+C_1}{x^2+b_1x+c_1} + \frac{B_2x+C_2}{(x^2+b_1x+c_1)^2} + \ldots + \frac{B_{m_1}x+C_{m_1}}{(x^2+b_1x+c_1)^{m_1}} +$$

$$+ \ldots$$

4. Gang med $Q(x)$, sett koeffisientene lik hverandre og finn $A_1, A_2, \ldots, A_{n_1}, \ldots B_1, C_1, \ldots$

5. Integrer:

grei

$$\int \frac{R(x)}{Q(x)}dx = \int \frac{A_1}{x-a_1}dx + \int \frac{A_2}{(x-a_1)^2}dx + \ldots + \int \frac{A_{n_1}}{(x-a_1)^{n_1}}dx$$

$$+ \ldots$$

$$+ \int \frac{B_1x+C_1}{x^2+b_1x+c_1} + \ldots$$

se nedenfor

forget it!

6: Integrasjon av $\int \frac{B_1x+C_1}{x^2+b_1x+c}dx$

a) Smugle den deriverte til nevneren inn i telleren og skift variabel $u =$ nevner.

b) Sitter igjen med

$\int \frac{D}{x^2+b_1x+c}dx$: Gjør kvadratet fullstendig og skift variabel.

Hva er vitsen?

1. Å integrere varierende funksjoner.
2. Slutt/få integrasjoner som kan omdannes til integrasjon av varierende funksjoner.

Eksempel: $\int 1 \cdot \ln(x^2+6x+10)\,dx$ $u=\ln(x^2+6x+10)$, $v'=1$

$u'=\dfrac{2x+6}{x^2+6x+10}$, $v=x$

$= x\ln(x^2+6x+10) - \int \dfrac{2x^2+6x}{x^2+6x+10}\,dx$

Hurra, delbrøksspalting!

$$\int \frac{2x^2+6x}{\underbrace{x^2+6x+10}_{2x^2+12x+20}}\,dx = \int \frac{2x^2+12x+20-6x-20}{x^2+6x+10}\,dx$$

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$$= \int \left(2 - \frac{6x+20}{x^2+6x+10}\right) dx = \ldots$$