Husk oblig-innlevering til fredag!

$16^{\underline{15}} - 18$

Noen integraler fra 9.4:

$$\int \sin^n x \cos^m x \, dx \quad , n, m \text{ hele tall}$$

Lette tilfellet: n, m (eller begge) odde:

Eksempel: $\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \overbrace{\cos^2 x}^{\text{like antall cosinus}} \cos x \, dx$

$u = \sin x \, , \; du = \cos x \, dx$

$= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx$

$= \int u^4 (1 - u^2) \, du = \int (u^4 - u^6) \, du = \frac{u^5}{5} - \frac{u^7}{7} + C$

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

Vanskelig tilfelle: Både n og m partall.

$$\cos 2u = \cos^2 u - \sin^2 u = 1 - 2\sin^2 u = 2\cos^2 u - 1$$

$$\int \sin^n x \cos^m x \, dx$$

$$\sin^2 u = \frac{1-\cos 2u}{2} \qquad \cos^2 u = \frac{1+\cos 2u}{2}$$

Eksempel: $\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \frac{1 - 2\cos 2x + \cos^2 2x}{4} \, dx$$

$$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{4}\int \cos^2 2x \, dx$$

$$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2} \, dx$$

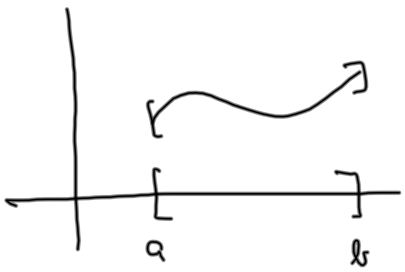
$$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x) \, dx$$

$$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{8}\cdot\frac{\sin 4x}{4} + C$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

9.5 Uegentlige integraler

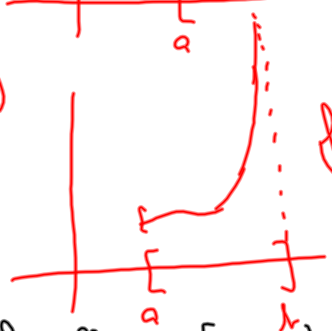
Hva har vi definert så langt? Integraler $\int_a^b f(x)\,dx$ der $a,b \in \mathbb{R}$ og f er begrenset.



Plan: Utvide til: $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$

a)

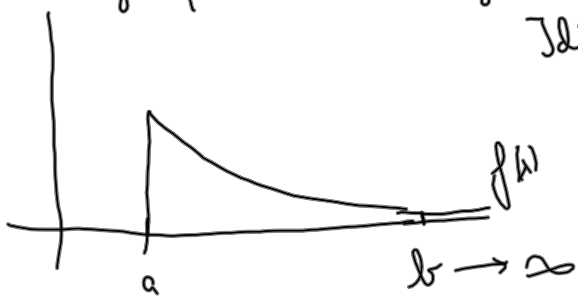
b)



$\int_a^b f(x)\,dx$ der $f(b)$ evt $f(a)$ ikke er definert.

a) Integrasjon over uendelige intervaller: $[a,\infty)$

Idé: $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$



<u>Definisjon:</u> Anta at $f:[a,\infty)\to\mathbb{R}$ er en kontinuerlig funksjon. Dersom

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

eksisterer, så sier vi at integralet $\int_a^\infty f(x)\,dx$ <u>konvergerer</u> og setter

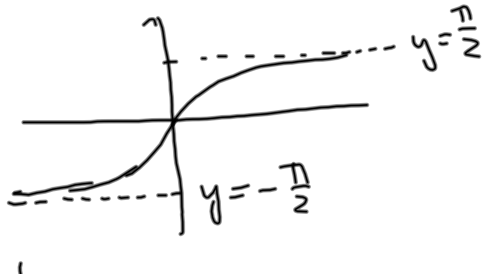
$$\int_a^\infty f(x)\,dx=\lim_{b\to\infty}\int_a^b f(x)\,dx$$

Dersom grenseverdien <u>ikke</u> eksisterer, sier vi at integralet $\int_a^\infty f(x)\,dx$ <u>divergerer</u>.

<u>Eksempel:</u> Avgjør om $\int_0^\infty \frac{1}{1+x^2}\,dx$ konvergerer og finn i så fall verdien.

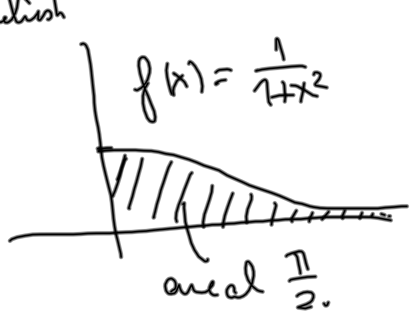
$$\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx=\lim_{b\to\infty}\Big[\arctan x\Big]_0^b=\lim_{b\to\infty}\Big[\arctan b-\overbrace{\arctan 0}^{0}\Big]$$

$$=\frac{\pi}{2}.$$



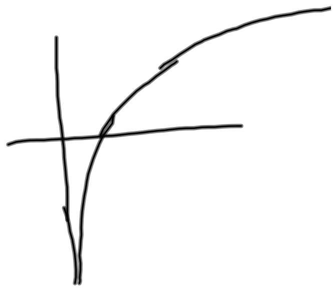
Altså konvergerer integralet, og

$$\int_0^\infty \frac{1}{1+x^2}\,dx=\frac{\pi}{2}$$ geometrisk



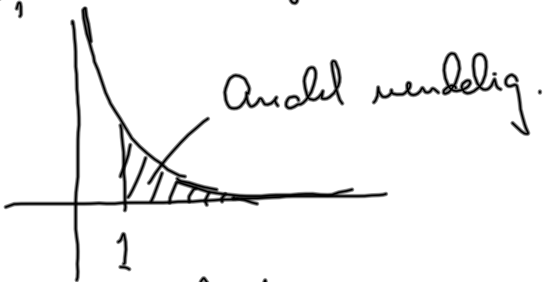
Eksempel: $\int_1^\infty \frac{1}{x}\,dx$ Ser på

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\Big[\ln|x|\Big]_1^b = \lim_{b\to\infty} \ln b = \infty$$



Integralet $\int_1^\infty \frac{1}{x}\,dx$ divergerer.

Geometrisk:



Setning: Integralet $\int_1^\infty \frac{1}{x^p}\,dx$ konvergerer når $p>1$ og divergerer når $p \leq 1$.

Bevis: Tilfellet $p=1$ har vi allerede sjekket. For $p \neq 1$, har vi

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{b^{-p+1}}{1-p} - \underline{\frac{1}{1-p}}\right] = \begin{cases} \infty & \text{når } -p+1>0 \\ -\frac{1}{1-p} & \text{når } -p+1<0 \end{cases}$$

$$= \begin{cases} \infty & \text{når } p<1 \quad \text{divergerer} \\ \frac{1}{p-1} & \text{når } p>1 \quad \text{konvergerer} \end{cases}$$

Grensesammenligningskriteriet: Anta at $f, g : [a, \infty) \to \mathbb{R}$ er to positive kontinuerlige funksjoner.

(i) Anta at $\int_a^\infty g(x)\,dx$ konvergerer og at $\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$, da konvergerer også $\int_a^\infty f(x)\,dx$

(ii) Anta at $\int_a^\infty g(x)\,dx$ divergerer og at $\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$, da divergerer også $\int_a^\infty f(x)\,dx$.

Bruk: Skal undersøke konvergens/divergens til $\int_a^\infty f(x)\,dx$, finner en enklere funksjon $g(x)$ å sammenligne med.

Eksempel: $\displaystyle\int_1^\infty \frac{x^3+2x^2+4}{x^4+4x^2+17}\,dx$ konvergerer/divergerer

$$f(x) = \frac{x^3+2x^2+4}{x^4+4x^2+17} = \frac{x^3\left(1+\frac{2}{x}+\frac{4}{x^3}\right)}{x^4\left(1+\frac{4}{x^2}+\frac{17}{x^4}\right)} = \frac{1}{x}\underbrace{\frac{\left(1+\frac{2}{x}+\frac{4}{x^3}\right)}{1+\frac{4}{x^2}+\frac{17}{x^4}}}_{\approx 1}$$

$f(x)$ oppfører seg omtrent som $g(x) = \frac{1}{x}$ for store x.

Vet at $\int_1^\infty g(x)\,dx$ divergerer.

Dersom $\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$, så vil også $\int_1^\infty f(x)\,dx$ divergere.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\cancel{\frac{1}{x}}\left(\frac{1+\frac{2}{x}+\frac{4}{x^3}}{1+\frac{4}{x^2}+\frac{17}{x^4}}\right)}{\cancel{\frac{1}{x}}} = 1 > 0$$

Konklusjon: $\int_1^\infty f(x)\,dx$ divergerer.

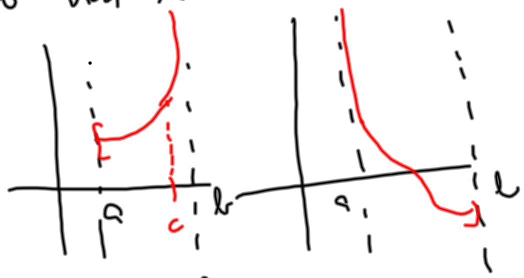
Uegentlige integraler av typen $\int_{-\infty}^{b} f(x)\,dx$ behandles på samme måte

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

< konvergerer hvis grensen fins
< divergerer ellers.

b) Integraler av typen

$$\int_a^b f(x)\,dx \quad \text{der } f(x)\to\infty \text{ når } x\to a \text{ eller } x\to b$$

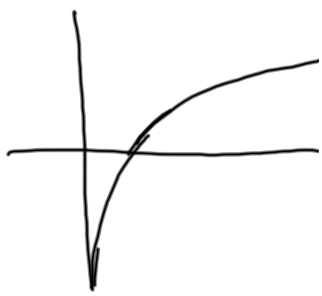


<u>Definisjon</u>: Anta at $f:[a,b)\to\mathbb{R}$ er kontinuerlig. Dersom

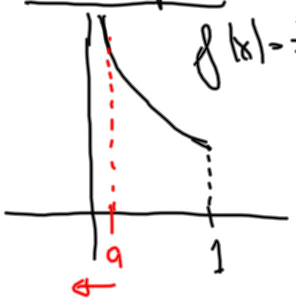
$\lim_{c\to b^-}\int_a^c f(x)\,dx$ eksisterer, sier vi at integralet $\int_a^b f(x)\,dx$

<u>konvergerer</u> og setter

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$



<u>Eksempel</u>: Konvergerer eller divergerer $\int_0^1 \frac{1}{x}\,dx$?

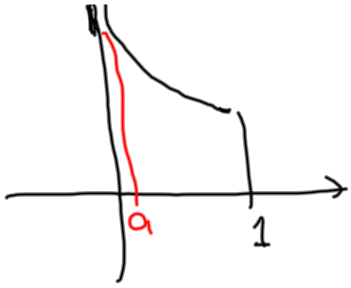


$$\lim_{a\to0^+}\int_a^1 \frac{1}{x}\,dx = \lim_{a\to0^+}\left[\ln x\right]_a^1$$

$$= \lim_{a\to0^+}\left[\underbrace{\ln 1}_{0} - \ln a\right] = -\lim_{a\to0^+}\ln a = \infty.$$

Divergerer!

Eksempel: $\int_0^1 \frac{1}{\sqrt{x}}\,dx$



$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[2\sqrt{x}\right]_a^1$$

$$= \lim_{a\to 0^+}\left(2\sqrt{1} - 2\sqrt{a}\right) = 2 \quad \text{Konvergens}$$

$\int_0^1 \frac{1}{x}\,dx$ divergerer

$\int_0^1 \frac{1}{x^{1/2}}\,dx$ konvergerer

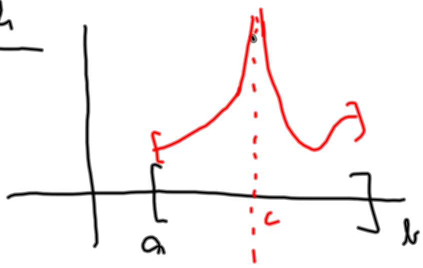
Setning: $\int_0^1 \frac{1}{x^p}\,dx$ konvergerer for $p<1$ og divergens for $p \geq 1$.

Utvidelse: $\int_{-\infty}^{\infty} f(x)\,dx$ konvergerer hvis $\int_a^{\infty} f(x)\,dx$ og $\int_{-\infty}^{a} f(x)\,dx$ begge konvergerer for en vilkårlig $a \in \mathbb{R}$.

I så fall setter vi

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx \,,$$

Tilsvarende



Integralet $\int_a^b f(x)\,dx$ konvergerer dersom $\int_a^c f(x)\,dx$ og $\int_c^b f(x)\,dx$ begge konvergerer og i så fall

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

FULA →