# Bruken av integrasjon

**Standardformler:** Areal: $\int_a^b f(x)\,dx$

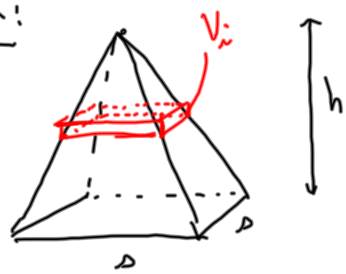
Omdreining om x-aksen: $\pi \int_a^b f(x)^2\,dx$

— " — y-aksen: $2\pi \int_a^b x f(x)\,dx$

Buelengden: $\int_a^b \sqrt{1+f'(x)^2}\,dx$

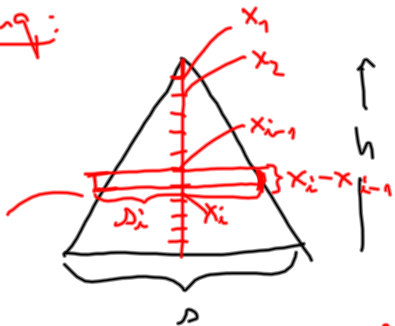
Viktigere: Å kunne sette opp integraler selv.

Eksempel:




Volum: $\frac{1}{3} gh = \frac{1}{3} s^2 h$

Regning:

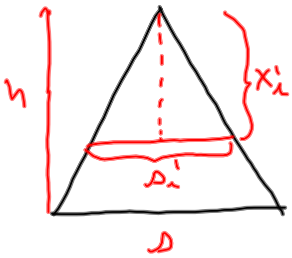



$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} s_i^2 (x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \left(\frac{s}{h} x_i\right)^2 (x_i - x_{i-1})$$

$$= \frac{s^2}{h^2} \underbrace{\sum_{i=1}^{n} x_i^2 (x_i - x_{i-1})}_{\text{Riemannsum for } f(x)=x^2} \longrightarrow \frac{s^2}{h^2}\int_0^h x^2\,dx$$

Hva er $s_i$:




Formlike: $\frac{s_i}{x_i} = \frac{s}{h}$

$s_i = \frac{s}{h} x_i$

Altså $V = \frac{s^2}{h^2}\int_0^h x^2\,dx = \frac{s^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{s^2}{\cancel{h^2}} \frac{h^{\cancel{3}}}{3} - 0 = \underline{\underline{\frac{s^2 h}{3}}}$

# Integrasjonsteknikk

**Tre grunnleggende teknikker:**

(i) Delvis integrasjon
(ii) Substitusjon (skifte av variabel)
(iii) Delbrøkoppspalting.

} Blande teknikken

## Delvis integrasjon

**Grunnleggende formel:** $\int uv'\,dx = uv - \int u'v\,dx$

**Utledning av formel:** Produktregel

$$(uv)' = u'v + uv'$$

Integrere

$$uv + C = \int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx$$

$$\int uv'\,dx = uv + C - \int u'v\,dx = uv - \int u'v\,dx$$

**Eksempel:** $\int \underbrace{x}_{u}\,\underbrace{\sin x}_{v'}\,dx$

$u = x \qquad v' = \sin x$
$u' = 1 \qquad v = -\cos x$

$$= \underbrace{-x\cos x}_{uv} - \int \underbrace{1}_{u'} \cdot \underbrace{(-\cos x)}_{v}\,dx = -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

**Eksempel:** $\int \underbrace{x^2}_{u}\,\underbrace{e^x}_{v'}\,dx =$

$u = x^2,\ v' = e^x$
$u' = 2x,\ v = e^x$

$u = 2x,\ v' = e^x$
$u' = 2,\ v = e^x$

$$= x^2e^x - \int 2xe^x\,dx$$

$$= x^2e^x - \left(2xe^x - \int 2e^x\,dx\right)$$

$$= x^2e^x - 2xe^x + 2\int e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

Funksjoner som blir mye enklere ved derivasjon er "fine" å ha i delvis integrasjon:

$u(x) = \ln x \, , \; u'(x) = \frac{1}{x}$

$u(x) = \arctan x \, , \; u'(x) = \frac{1}{1+x^2}$

$u(x) = \arcsin x \, , \; u'(x) = \frac{1}{\sqrt{1-x^2}}$

<u>Eksempel:</u> $\int \underbrace{x^3}_{v'} \underbrace{\ln x}_{u} \, dx$

$u = \ln x \, , \; v' = x^3$

$u' = \frac{1}{x} \, , \; v = \frac{x^4}{4}$

$$= \frac{x^4}{4}\ln x - \int \frac{1}{x}\,\frac{x^4}{4}\,dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3\,dx = \frac{x^4}{4}\ln x - \frac{1}{4}\left[\frac{x^4}{4}\right] + C$$

$$= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C$$

Eksempel: $\int \underbrace{x}_{v'} \underbrace{\arctan x}_{u} \, dx$ $u = \arctan x$ $v' = x$

$u' = \frac{1}{1+x^2}$, $v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\arctan x - \int \frac{\frac{x^2}{2}}{1+x^2}\, dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + C$$

Mellomregning:

$$\int \frac{x^2}{1+x^2}\, dx = \int \frac{(1+x^2)-1}{1+x^2}\, dx$$

$$= \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \arctan x + C$$

Eksempel: $\int \arcsin x \, dx = \int \underbrace{1}_{v'} \cdot \underbrace{\arcsin x}_{u} \, dx$

$u = \arcsin x$, $v' = 1$

$u' = \frac{1}{\sqrt{1-x^2}}$, $v = x$

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx.$$

$z = 1-x^2$

$dz = -2x\, dx$

$x\, dx = -\frac{1}{2}dz$

$$= x\arcsin x - \int \frac{\left(-\frac{1}{2}dz\right)}{\sqrt{z}}$$

$\sqrt{z} = z^{1/2}$

$$= x\arcsin x + \int \frac{1}{2\sqrt{z}}\, dz = x\arcsin x + \sqrt{z} + C$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

Annen bruk av delvis integrasjon: Tilbake til utgangspunktet (nesten).

Eksempel: $\int \sin^2 x\, dx$

$u = \sin x$, $v' = \sin x$

$u' = \cos x$, $v = -\cos x$

$= -\sin x \cos x + \int \cos^2 x\, dx$

$= -\sin x \cos x + \int (1 - \sin^2 x)\, dx$

$= -\sin x \cos x + x - \int \sin^2 x\, dx$

dvs: $\underbrace{\int \sin^2 x\, dx} = -\sin x \cos x + \overset{C}{x} - \underbrace{\int \sin^2 x\, dx}$

$2\int \sin^2 x\, dx = -\sin x \cos x + x$

$\int \sin^2 x\, dx = -\frac{1}{2}\sin x \cos x + \frac{x}{2} + C$

Iterasjonsformler:

Ønsker å regne ut $\int x^{13} e^x\, dx$.

$u = x^{13}$, $v' = e^x$

$u' = 13x^{12}$, $v = e^x$

$\int x^{13} e^x\, dx = x^{13} e^x - 13\int x^{12} e^x\, dx$

Generell vedutsjon: $I_n = \int x^n e^x\, dx$

$u = x^n$, $v' = e^x$

$u' = nx^{n-1}$, $v = e^x$

$= x^n e^x - \int n x^{n-1} e^x\, dx = x^n e^x - n\underbrace{\int x^{n-1} e^x\, dx}_{I_{n-1}} = x^n e^x - nI_{n-1}$

Iterasjonsformel: $I_n = x^n e^x - nI_{n-1}$

Regner ut $I_3$: $I_3 = x^3 e^x - 3I_2 = x^3 e^x - 3(x^2 e^x - 2I_1)$

$= x^3 e^x - 3x^2 e^x + 6I_1 = x^3 e^x - 3x^2 e^x + 6(x e^x - 1 \cdot I_0)$

$= x^3 e^x - 3x^2 e^x + 6x e^x - 6\int \underbrace{x^0}_{1} e^x\, dx = \underline{x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C}$

$-6\int e^x\, dx$

## Substitusjon (9.2)

Har sett: $\int f(\underbrace{g(x)}_{u=g(x)})\underbrace{g'(x)\,dx}_{du=g'(x)dx} = \int f(u)\,du \Big|_{u=g(x)}$

Men hva hvis: $\int f(g(x))\,dx$ der vi har lyst til å sette $u=g(x)$

Anta at $u=g(x)$ er injektiv og deriverbar med $g'(x)\neq 0$.
La $x=h(u)$ være den omvendte funksjonen. Vet

$h'(g(x)) = \cancel{h'(u)} = \frac{1}{g'(x)} \Rightarrow \underbrace{h'(g(x))g'(x)}=1$. Dermed

$\int f(g(x))dx = \int \underbrace{f(g(x))h'(g(x))}\underbrace{g'(x)\,dx} = $ $u=g(x)$, $du=g'(x)dx$

$= \int f(u)\underbrace{h'(u)}\,du \Big|_{u=g(x)}$

Formel

$$\int f(g(x))\,dx = \int f(u)h'(u)\,du\Big|_{u=g(x)}$$

og der $h$ er den omvendte funksjonen til $g$.

I praksis: $\int f(g(x))\,dx$  $u=g(x) \Downarrow x=h(u)$

$= \int f(u)h'(u)\,du \Big|_{u=g(x)}$  $\frac{dx}{du}=h'(u) \Rightarrow dx=h'(u)du$

Eksempel: $I=\int e^{\sqrt{x}}\,dx$  $u=\sqrt{x} \Rightarrow x=u^2,\ dx=2u\,du$

$= \int e^u\,2u\,du = \int 2ue^u\,du$

$$\boxed{\begin{array}{l} U=2u,\ V'=e^u \\ U'=2,\ V=e^u \end{array}}$$

$= 2ue^u - \int 2e^u\,du$

$= 2ue^u - 2e^u + C =$

$= \underline{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$

Eksempel: $I = \int \sqrt{4 - \underbrace{x^2}_{4\sin^2 x}}\, dx$ $\qquad x = 2\sin u \Rightarrow \frac{x}{2} = \sin u$

$\Rightarrow u = \arcsin \frac{x}{2}$

$= \int \sqrt{4 - 4\sin^2 u}\; 2\cos u\, du \qquad dx = 2\cos u\, du$

$= \int 2 \underbrace{\sqrt{1 - \sin^2 u}}_{\cos u} \cdot 2\cos u\, du = 4\int \cos^2 u\, du$

$U = \cos u, \; V' = \cos u$

Mellomregning: $\int \cos^2 u\, du$ $\qquad U' = -\sin u, \; V = \sin u$

$= \cos u \sin u + \int \sin^2 u\, du$

$= \cos u \sin u + \int (1 - \cos^2 u)\, du = \underbrace{\cos u \sin u + u}_{} - \int \cos^2 u\, du$

altså: $\int \cos^2 u\, du = \frac{1}{2} \cos u \sin u + \frac{u}{2} + C$

$x = 2\sin u$

Tilbake: $I = 4\int \cos^2 u\, du = 2\cos u \sin u + 2u + C$ $\qquad u = \arcsin \frac{x}{2}$

$= 2\sqrt{1 - \sin^2 u}\, \sin u + 2u + C$

$= 2\sqrt{1 - \frac{x^2}{4}} \cdot \frac{x}{2} + 2\arcsin \frac{x}{2} + C$