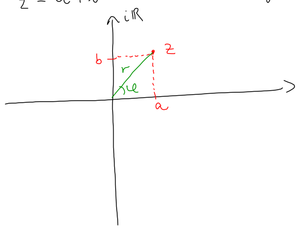
4/8/14: Komplekse tall

$z = a + ib$ der $i = \sqrt{-1}$ og $a, b \in \mathbb{R}$.



	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	"∞"	0

sep 4-08:08

Oppg. 1

(c) $2i + 3(4+i) = 2i + 12 + 3 = 12 + 5i$

(f) $\frac{4+3i}{2+i} \cdot \frac{2-i}{2-i} = \frac{8-4i+6i-3i^2}{4-\cancel{2i}+\cancel{2i}-i^2} = \frac{8+2i+3}{5} = \frac{11}{5} + \frac{2}{5} \cdot i$

$i^2 = (-1)$

realdel imaginærdel

(h) $(5-i)^{-1} = \frac{1}{5-i} \cdot \frac{5+i}{5+i} = \frac{5+i}{5^2+1} = \frac{5+i}{26} = \frac{5}{26} + i\frac{1}{26}$

Husk: $z = a + i \cdot b$ da er $\bar{z} = a - i \cdot b$ (konjugert)

$\hookrightarrow z \cdot \bar{z} = |z|^2$.

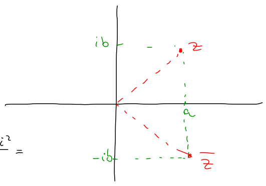
sep 4-08:20

Oppg. 3 Regn ut:

(b) $\overline{2+3i} = 2 - 3i$.

(f) $(2-3i) + i \cdot \frac{4+5i}{1-i}$

(**) (*)



$(*) = \frac{4+5i}{1-i} \cdot \frac{1+i}{1+i} = \frac{4+4i+5i+5i^2}{2} =$

$= \frac{-1+9i}{2} = -\frac{1}{2} + \frac{9}{2}i$

$(**) = (2-3i) + i\left(-\frac{1}{2} + \frac{9}{2}i\right) = 2 - 3i - \frac{1}{2}i - \frac{9}{2} = -\frac{5}{2} - \frac{7}{2}i$

(f) $\overline{(2-3i) + i\frac{4+5i}{1-i}} = \overline{-\frac{5}{2} - \frac{7}{2}i} = -\frac{5}{2} + \frac{7}{2}i$

sep 4-08:26

Oppg 5 Løs ligningene

(C) Finne z s.a. $\frac{z-2}{z+1} = 3i \iff z - 2 = 3i(z+1) \iff$

$z - 2 = 3i \cdot z + 3i \iff z - 3i \cdot z = 2 + 3i$

$z(1-3i) = 2 + 3i \iff z = \frac{2+3i}{1-3i} \cdot \frac{1+3i}{1+3i} = \dots = -\frac{7}{10} + \frac{9}{10}i$

Oppg 7 Finn komplekse z og w s.a.

$$\left.\begin{aligned} z - w &= 1 - i \quad (*) \\ + \quad z + w &= 4 \end{aligned}\right\} \text{plusse ligningene:}$$

$2z = 5 - i \rightsquigarrow \boxed{z = \frac{5}{2} - \frac{1}{2}i} \rightsquigarrow \overset{(*)}{} \boxed{w = z - 1 + i = \frac{3}{2} + \frac{1}{2}i}$

sep 4-08:32

Oppg 8 Bevise regnereglene for konjugasjon.

(ii) $\overline{z - w} = \bar{z} - \bar{w}$

$a, b \in \mathbb{R}$ $z = a + ib \longrightarrow \bar{z} = a - ib$

$c, d \in \mathbb{R}$ $w = c + id \longrightarrow \bar{w} = c - id$.

$\overline{z - w} = \overline{a + ib - (c + id)} = \overline{a - c + i \cdot (b - d)} = a - c - i \cdot (b - d)$

$= a - ib - c + id = a - ib - (c - id) = \bar{z} - \bar{w}$.

(iv) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

$\hookrightarrow \overline{\left(\frac{a+ib}{c+id}\right)} = \overline{\left(\frac{a+ib}{c+id} \cdot \frac{c-id}{c-id}\right)} = \overline{\frac{ac - iad + ibc + bd}{c^2+d^2}}$

$= \overline{\frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}} = \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2}$

$\frac{\bar{z}}{\bar{w}} = \frac{a-ib}{c-id} \cdot \frac{c+id}{c+id} = \frac{ac + iad - ibc + bd}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2}$

Oppg. 10

sep 4-08:38

Oppg 10 Anta at både $z + w$ og zw er reelle tall. Vis at

enten z og w er reelle eller $z = \bar{w}$.

$z = a + ib$ $\quad z + w = a + c + (b+d)i \in \mathbb{R}$ $\Longrightarrow$ $b + d = 0$ (I)

$w = c + id$ $\quad z \cdot w = (ac - bd) + i(bc + ad) \in \mathbb{R}$ $\quad bc + ad = 0$ (II)

Ganger (I) med $-c \neq 0$.

$$\begin{aligned} -bc - cd &= 0 \\ + \quad bc + ad &= 0 \\ \hline ad - cd &= 0 \end{aligned} \quad (*) \quad \xrightarrow{d \neq 0} \boxed{a = c}$$

Fra (I) $b = -d$

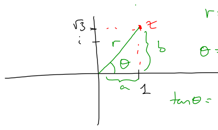
og fra (*) $a = c$ dvs. $z = a + ib$, $w = \underset{c}{a} - \underset{id}{ib} \Rightarrow \bar{w} = z$

Når $d = 0 \Rightarrow b = 0$ fra (I) $z, w \in \mathbb{R}$

Seksjon 3.2: Geometrisk tolkning av komplekse tall

Oppg 3: Finn modulus r og argument θ til disse tallene

(e) $z = 1 + i \cdot \sqrt{3}$



$r = \sqrt{a^2 + b^2} = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$.

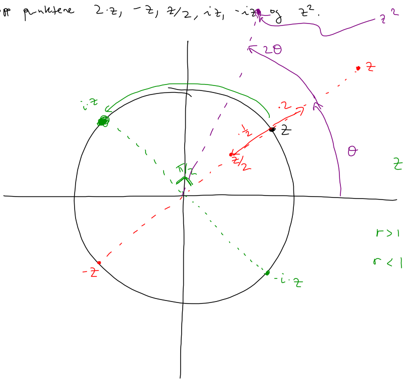
$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(\sqrt{3}) =$

$= \pi/3$ $(60°)$

$\tan\theta = \frac{b}{a}$.

sep 4-08:43

Oppg 8: Tegn det komplekse tallplanet og merk av et punkt z. Tegn deretter opp punktene $2\cdot z$, $-z$, $z/2$, iz, $-iz$ og z^2.



$i\cdot z = \underbrace{e^{i\frac{\pi}{2}}}_{\text{rotasjon med vinkel } \pi/2}\cdot z$

$z = re^{i\theta}$

$z^2 = r^2\cdot e^{i\cdot 2\theta}$

$r>1 \rightsquigarrow r^2 >> 1 \quad 2\theta > \theta.$

$r<1 \rightsquigarrow r^2 << 1 \quad 2\theta > \theta.$

sep 4-09:13

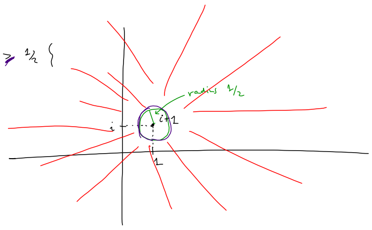
Oppg 20 Skissér følgende områder i det komplekse planet.

(c) $C = \{z \mid |z-(i+1)| \geq 1\}$ (d) $D = \{z \mid |z-2| < |z-i+2|\}$

$B = \{z \mid |z-w| \leq r\}$ der w er et bestemt kompleks tall

avstanden mellom z og w.

$C = \{z \mid |z-(i+1)| \geq 1/2\}$



sep 4-09:25

(d) $D = \{z \mid |z-2| < |z-i+2|\} = \{z \mid |z-2| < |z-(i-2)|\}.$

avstanden mellom z og 2 — avstanden mellom z og i−2.

$z = a+ib$, $a,b \in \mathbb{R}$

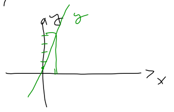
$|z-2| = |a-2+ib| = \sqrt{(a-2)^2+b^2}$

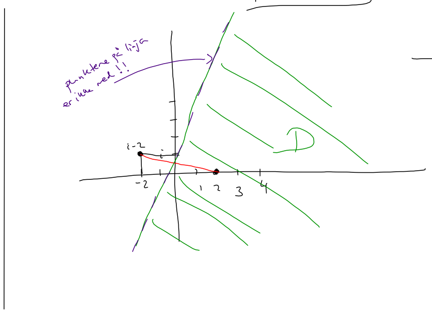
$|z-(i-2)| = |a+2+i(b-1)| = \sqrt{(a+2)^2+(b-1)^2}$

$\Leftrightarrow (a-2)^2+b^2 = (a+2)^2+(b-1)^2$

$2b = 8a+1 \iff \boxed{b = 4a + 1/2}$

$y = 4x + 1/2$





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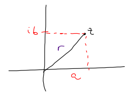
Oppg 13 $z = 1+i\sqrt{3}$, $w = 1+i$.

(a) $z\cdot w = (1+i\sqrt{3})(1+i) = 1+i+i\sqrt{3}+i^2\sqrt{3} = 1-\sqrt{3}+i(1+\sqrt{3})$

(b) $z = \text{"modulus"}\cdot e^{i\text{"vinkel"}} = re^{i\theta}$



$r = \sqrt{a^2+b^2}$

$w = s\cdot e^{i\varphi}$

$r = \sqrt{1^2+\sqrt{3}^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}(\sqrt{3}) = \pi/3 \longrightarrow \boxed{z = 2\cdot e^{i\pi/3}}$

$s = \sqrt{1^2+1^2} = \sqrt{2}$

$\varphi = \tan^{-1}(1) = \pi/4 \longrightarrow \boxed{w = \sqrt{2}\, e^{i\pi/4}}$

(c) $\frac{z}{w} = \frac{re^{i\theta}}{se^{i\varphi}} = \frac{r}{s}\cdot e^{i(\theta-\varphi)} = \sqrt{2}\cdot e^{i(\pi/3-\pi/4)} = \sqrt{2}\, e^{i\pi/12}$

Oppg 15

Vis at $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$, $z,w\in\mathbb{C}$

$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w}) = \underbrace{z\cdot\bar{z}}_{|z|^2} + z\cdot\bar{w} + w\cdot\bar{z} + \underbrace{w\cdot\bar{w}}_{|w|^2}$ +

$|z-w|^2 = (z-w)\overline{(z-w)} = (z-w)\cdot(\bar{z}-\bar{w}) = \underbrace{z\cdot\bar{z}}_{|z|^2} - z\cdot\bar{w} - w\cdot\bar{z} + \underbrace{w\cdot\bar{w}}_{|w|^2}$

$2|z|^2$ / / $2|w|^2$

$\rightarrow \boxed{|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2}$

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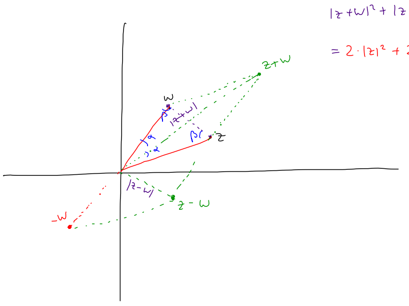
$\boxed{|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2}$



$|z+w|^2 + |z-w|^2 =$

$= 2\cdot|z|^2 + 2|w|^2$

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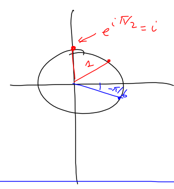
Seksjon 3.3:

Oppg 1 (a) $e^{i\pi/2}$

$e^{i\pi/2} = i$ $\boxed{a=0 \quad b=1}$



(c) $e^{-i\pi/6} = r\cos(\theta) + i\,r\sin(\theta)$

$= 1\cdot\cos(-\pi/6) + i\,1\cdot\sin(-\pi/6)$

$= \cos(\pi/6) - i\sin(\pi/6)$

$= \frac{\sqrt{3}}{2} - i\,\frac{1}{2}$

$\boxed{z = re^{i\theta} = r\cdot\cos\theta + i\cdot r\cdot\sin\theta}$

sep 4-09:57