Integrasjon

Analysens fundamentalteorem (kalkulusen): Hvis f er en kontinuerlig funksjon, så er $F(x) = \int_a^x f(t)\,dt$ en antiderivert til f, dvs $F'(x) = f(x)$.

Eksempel: Den deriverte til $F(x) = \int_0^x e^{-t^2}\,dt$, er $F'(x) = e^{-x^2}$.

Hva med $G(x) = \int_0^{x^2} e^{-t^2}\,dt$? Hva er da $G'(x)$?

Generelt: $G(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, hva er da $G'(x)$?

Hvis F er en antiderivert til f, så er

$$G(x) = \int_{g(x)}^{h(x)} f(t)\,dt = F(h(x)) - F(g(x))$$

$$G'(x) = F'(h(x))h'(x) - F'(g(x))g'(x) = \underbrace{f(h(x))h'(x)} - \underline{f(g(x))g'(x)}.$$

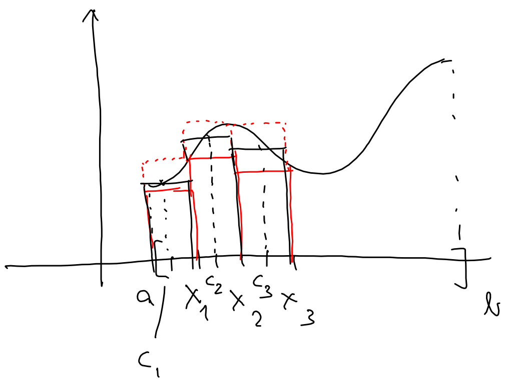
Eksempel.
$$\lim_{x\to 0} \frac{\int_0^{x^2} \frac{\sin t}{t+1}\,dt}{1-\cos x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{\frac{\sin x^2}{x^2+1}2x}{\sin x} = \underbrace{\lim_{x\to 0} \frac{2\sin x^2}{x^2+1}} \underbrace{\lim_{x\to 0}\frac{x}{\sin x}}$$

Mellomregning: $\left(\int_0^{x^2} \frac{\sin t}{t+1}\,dt\right)' = \frac{\sin x^2}{x^2+1}2x$

$$= 0 \cdot 1 = \underline{\underline{0}}$$

Riemannsummer

Øvre og nedre trappesummer:



Partisjon:

$$\Pi = \{x_0, x_1, x_2, \ldots, x_n\}$$

$x_0 = a$, $x_n = b$

Utplukk U består av ett punkt fra hvert delintervall:

$$U = \{c_1, c_2, \ldots, c_n\}$$

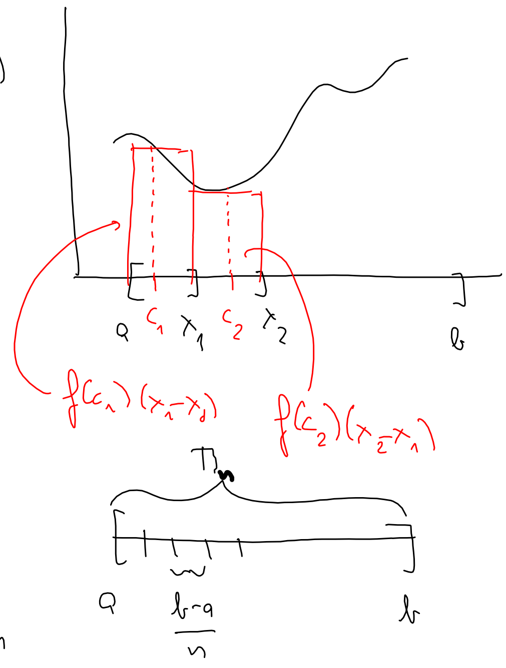
$c_1 \in [x_0, x_1]$, $c_2 \in [x_1, x_2], \ldots$

Riemannsummen til Π og U:

$$R(\Pi, U) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

Maskevidden til Π:

$$|\Pi| = \max\{x_i - x_{i-1} : i = 1, \ldots, n\}$$



Teorem: Anta at $\{\Pi_n\}$ er en følge av partisjoner slik at $|\Pi_n| \to 0$ når $n \to \infty$. Hvis U_n er et utplukk for Π_n, så vil Riemannsummene til en integrerbar funksjon f konvergere mot integralet, dvs

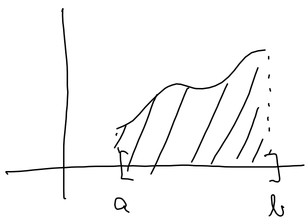
$$R(\Pi_n, U_n) \longrightarrow \int_a^b f(x)\,dx$$

$$\sum f(c_i)(x_i - x_{i-1}) \to \int_a^b f(x)\,dx$$

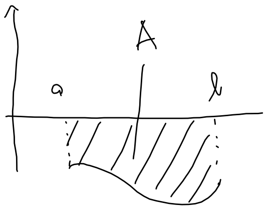
Anvendelser av integralet

Arealer

Hvis $f \geq 0$ er en integrerbar funksjon, så er arealet under funksjonsgrafen mellom $x=a$ og $x=b$ gitt ved



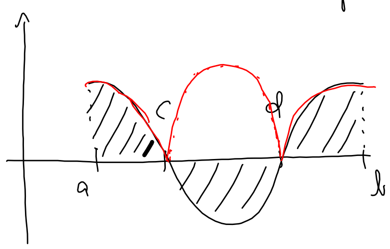
$$A = \int_a^b f(x)\,dx$$



Hvis $f \leq 0$, så er

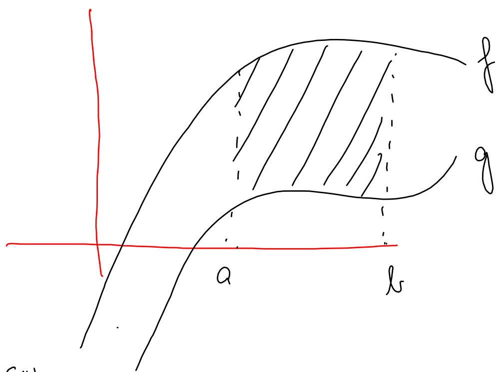
$$A = -\int_a^b f(x)\,dx.$$

Skiftende fortegn:



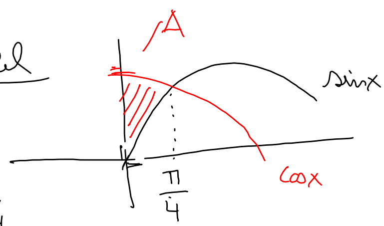
$$A = \int_a^c f(x)\,dx - \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$$

$$= \int_a^b |f(x)|\,dx$$



$$A = \int_a^b [f(x) - g(x)]\,dx \qquad f \geq g$$

Eksempel



Søker:

$\sin x = \cos x$

$\Downarrow$

$\tan x = 1$

$\Downarrow$

$x = \frac{\pi}{4}$

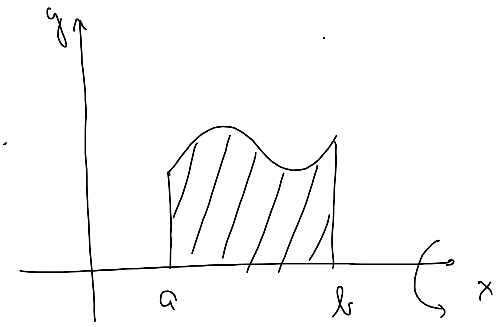
$$A = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4}$$

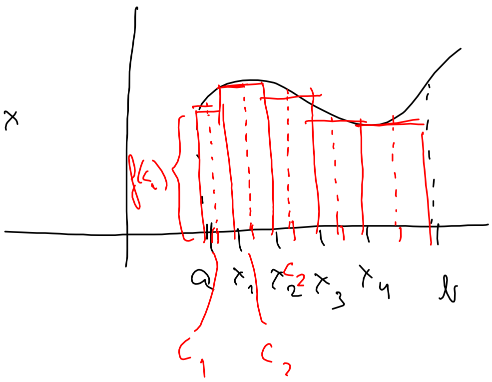
$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2} - 1$$

Volumer

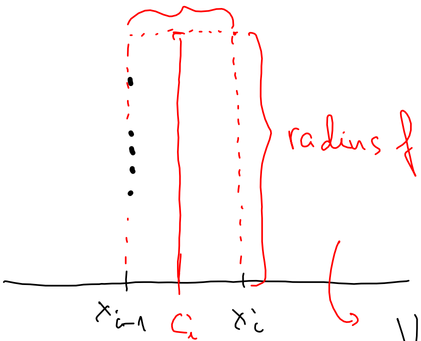
Omdreiningslegeme rundt x-aksen.



$$V = \pi \int_a^b f(x)^2\,dx$$



høyde = $x_i - x_{i-1}$



radius $f(c_i)$ $\quad V_i = \pi f(c_i)^2 (x_i - x_{i-1})$

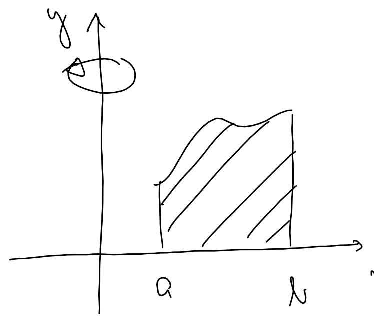
Tilnærmet verdi for hele volumet:

$$V \longleftarrow \sum_{i=1}^{n} V_i = \underbrace{\sum_{i=1}^{n} \pi f(c_i)^2 (x_i - x_{i-1})}_{\text{Riemannsum for } \pi \cdot f(x)^2} \longrightarrow \int_a^b \pi f(x)^2\,dx$$

Konklusjon:

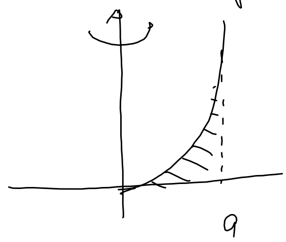
$$V = \pi \int_a^b f(x)^2\,dx$$

Omdreiningslegeme rundt y-aksen



$$V = 2\pi \int_a^b x\, f(x)\, dx$$

Eksempel: $f(x) = x^2, \quad 0 \le x \le a$

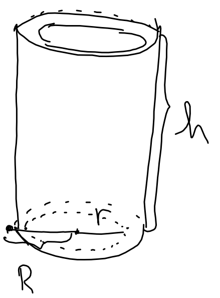


$$V = 2\pi \int_0^a x \cdot x^2\, dx$$

$$= 2\pi \int_0^a x^3\, dx = 2\pi \left[\frac{x^4}{4}\right]_0^a$$

$$= 2\pi \frac{a^4}{4} = \underline{\underline{\frac{\pi a^4}{2}}}$$

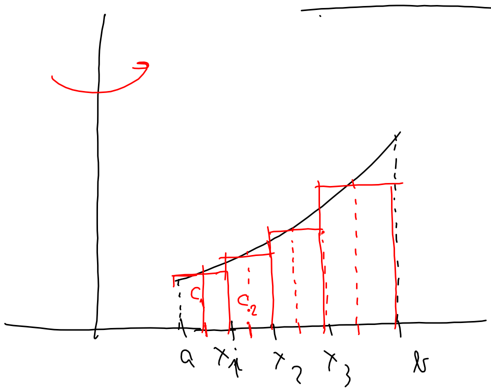
Forpostfektning: Volum til rør:

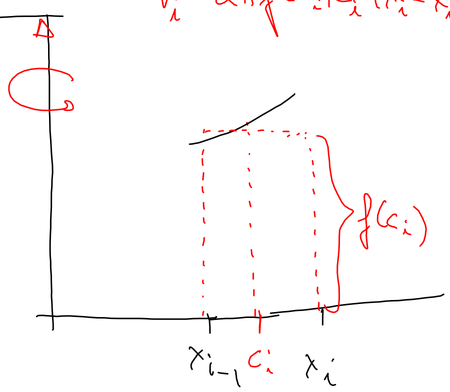


$$V_{rør} = \pi R^2 h - \pi r^2 h = \pi h (R+r)(R-r)$$

$$= 2\pi h \underbrace{\frac{R+r}{2}}_{\substack{\text{middel}\\ \text{radius}\\ r^*}} \underbrace{(R-r)}_{\substack{\text{radiedifferanse}\\ \Delta r}} = \underline{2\pi h r^* \Delta r}$$

$$V_i = 2\pi f(c_i) c_i (x_i - x_{i-1})$$



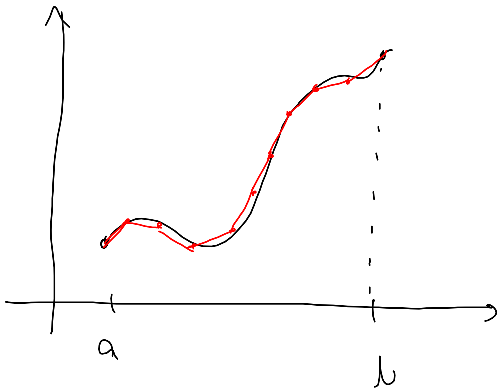


$$V \longleftarrow \sum_{i=1}^n V_i = \underbrace{\sum_{i=1}^n 2\pi c_i f(c_i)(x_i - x_{i-1})}_{\substack{\text{Riemannsum til}\\ 2\pi x f(x)}} \longrightarrow \int_a^b 2\pi x f(x)\, dx$$

Altså:

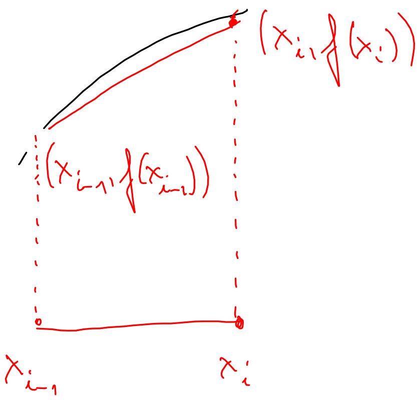
$$V = 2\pi \int_a^b x\, f(x)\, dx$$

Buelengde



Hvor lang er funksjonsgrafen?

Regner ut lengden av den brudne kurven isteden.



$$l_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

Tilnærmet lengde

$$\sum l_i = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \underbrace{\left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}_{f'(c_i)^2}}\ (x_i - x_{i-1}) \quad \text{der } c_i \in (x_{i-1}, x_i)$$

$$= \underbrace{\sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\ (x_i - x_{i-1})}_{\text{Riemannsum for } \sqrt{1+f'(x)^2}} \longrightarrow \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Buelengden til funksjonsgrafen $y = f(x)$ fra $x = a$ til $x = b$ er lik:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$