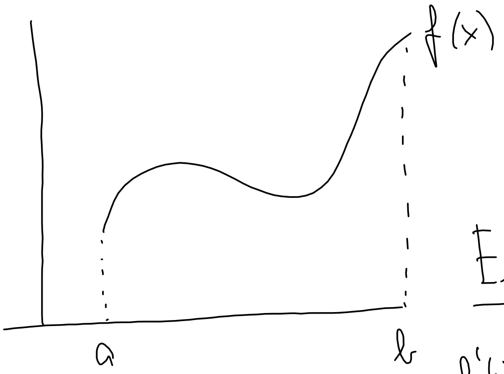
Buelengde



Lengden til funksjonsgrafen er

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Eksempel: $f(x) = \frac{1}{2}\ln x - \frac{x^2}{4}, \quad 1 \le x \le e$

$$f'(x) = \frac{1}{2x} - \frac{x}{2}$$

$$L = \int_1^e \sqrt{1 + \left(\frac{1}{2x} - \frac{x}{2}\right)^2}\, dx = \int_1^e \sqrt{1 + \frac{1}{4x^2} - \frac{1}{2} + \frac{x^2}{4}}\, dx$$

$$= \int_1^e \sqrt{\frac{1}{4x^2} + \frac{1}{2} + \frac{x^2}{4}}\, dx = \int_1^e \sqrt{\left(\frac{1}{2x} + \frac{x}{2}\right)^2}\, dx = \int_1^e \left(\frac{1}{2x} + \frac{x}{2}\right) dx$$

$$= \left[\frac{1}{2}\ln|x| + \frac{x^2}{4}\right]_1^e = \left(\underset{\frac{1}{2}}{\frac{1}{2}}\,\underbrace{\ln e}_{1} + \frac{e^2}{4}\right) - \left(\frac{1}{2}\,\underbrace{\cancel{\ln 1}}_{0} + \underset{1/4}{\frac{1^2}{4}}\right) = \underline{\underline{\frac{e^2}{4} + \frac{1}{4}}}$$

Integrasjonsteknikker

Delvis integrasjon
Substitusjon
Delbrøksoppspaltning

} Bygger ut teknikkene.

Delvis integrasjon

omformer $\int uv'\,dx$ til $\underbrace{\int u'v\,dx}_{\text{enklere}}$

Formel: $\int uv'\,dx = uv - \int u'v\,dx$

Hvor kommer formelen fra: Produktregel

$(uv)' = u'v + uv'$

Integrerer:

$$uv + C = \cancel{\int (uv)'dx} = \underbrace{\int u'v\,dx}_{\text{Flytter over}} + \underbrace{\int uv'\,dx} \Rightarrow \int uv'\,dx = uv - \int u'v\,dx \cancel{+C}.$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Eksempel: $\int \underbrace{x}_{u}\,\underbrace{\sin x}_{v'}\,dx$ $u = x$ $v' = \sin x$, $u' = 1$ $v = -\cos x$

$$= -x\cos x - \int 1\cdot(-\cos x)\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

Eksempel: $\int \underbrace{x^2}_{u}\underbrace{e^x}_{v'}\,dx$ $u = x^2$ $v' = e^x$, $u' = 2x$ $v = e^x$

$$= x^2e^x - \int 2xe^x\,dx$$ $u = 2x$ $v' = e^x$, $u' = 2$ $v = e^x$

$$= x^2e^x - \left[2xe^x - \int 2e^x\,dx\right] = x^2e^x - 2xe^x + 2\int e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

Funksjoner som blir enklere ved derivasjon: $u = \ln|x|$, $u' = \frac{1}{x}$

$u = \arctan x$, $u' = \frac{1}{1+x^2}$

$u = \arcsin x$, $u' = \frac{1}{\sqrt{1-x^2}}$

Eksempel: $\int \underbrace{x}_{v'}\,\underbrace{\ln x}_{u}\,dx$ $u = \ln x$, $v' = x$, $u' = \frac{1}{x}$, $v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

Eksempel: $\int \arctan x \, dx = \int \underbrace{1}_{v'} \cdot \underbrace{\arctan x}_{u} \, dx$

$u = \arctan x$, $v' = 1$

$u' = \frac{1}{1+x^2}$, $v = x$

$= x \arctan x - \int \frac{1}{1+x^2} \cdot \underbrace{x \, dx}$

Substitusjon: $u = 1+x^2$

$du = 2x \, dx$

$x \, dx = \frac{1}{2} du$

$= x \arctan x - \int \frac{1}{u} \frac{1}{2} du$

$= x \arctan x - \frac{1}{2} \ln|u| + C = \underline{x \arctan x - \frac{1}{2} \ln(1+x^2) + C}$

En annen variant: (syklisk)

Eksempel: $\int \underbrace{e^x}_{v'} \underbrace{\sin x}_{u} \, dx = e^x \sin x - \int \underbrace{e^x}_{v'} \underbrace{\cos x}_{u} \, dx$

$u = \sin x$ $v' = e^x$

$u' = \cos x$ $v = e^x$

$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$

$u = \cos x$ $v' = e^x$

$u' = -\sin x$ $v = e^x$

$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

Altså

$\underbrace{\int e^x \sin x \, dx} = e^x \sin x - e^x \cos x \underbrace{- \int e^x \sin x \, dx}$

$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$

$\int e^x \sin x \, dx = \underline{\frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C}$

Systematisering

$$\int x^4 e^x \, dx \qquad u = x^4,\ v' = e^x$$

$$= x^4 e^x - 4\int x^3 e^x \, dx \qquad u' = 4x^3,\ v = e^x$$

$$\vdots \qquad u = x^3,\ v' = e^x$$

$$u' = 3x^2,\ v = e^x$$

Kjedelig!

$$I_n = \int x^n e^x \, dx \qquad u = x^n,\ v' = e^x \qquad u' = nx^{n-1},\ v = e^x$$

$$= x^n e^x - \int n x^{n-1} e^x \, dx = x^n e^x - n\underbrace{\int x^{n-1} e^x \, dx}_{I_{n-1}} = x^n e^x - nI_{n-1}$$

Rekursjonsformel:

$$\boxed{I_n = x^n e^x - nI_{n-1}}$$

Eksempel: $I_4 = x^4 e^x - 4I_3 = x^4 e^x - 4(x^3 e^x - 3I_2)$

$$= x^4 e^x - 4x^3 e^x + 12I_2 = x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2\overset{}{I_1})$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24(\overbrace{xe^x - 1\cdot I_0}^{I_1})$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24I_0$$

$$= \underline{x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C}$$

$$I_0 = \int x^0 e^x \, dx = \int e^x \, dx = e^x + C$$

Substitusjon (skifte av variabel)

Grunnleggende teknikk: $\int f(g(x))g'(x)\,dx$ $\qquad u=g(x)$, $du=g'(x)\,dx$

$$= \int f(u)\,du = F(u)+C = F(g(x))+C$$

Flaks!

Hva gjør vi når den magiske faktoren $g'(x)$ mangler?

$\int f(g(x))\,dx$ - smugler inn $g'(x)$!

Anta at g har en omvendt funksjon h. Da er

$$h'(g(x)) = \frac{1}{g'(x)} \Rightarrow h'(g(x))g'(x) = 1$$

Dermed

$$\int f(g(x))\overset{1}{\vee}\,dx = \int f(g(x))\overset{\text{Pris for magisk faktor}}{h'(g(x))}\underbrace{g'(x)\,dx}_{\text{magisk faktor}} \qquad u=g(x),\ du=g'(x)\,dx$$

$$= \int f(u)h'(u)\,du$$

Altså

$$\int f(g(x))\,dx = \int f(u)h'(u)\,du \qquad \text{der } u=g(x),\ h=g^{-1}$$

Praktisk:

$$\int f(g(x))\,dx = \int f(u)h'(u)\,du$$

Setter $u=g(x) \Rightarrow x=h(u)$ (løser for x)

$\frac{dx}{du} = h'(u) \Rightarrow dx = h'(u)\,du$

Eksempel: $\int \frac{1}{\sqrt[3]{x}+1}\,dx$

løser for x: $u=\sqrt[3]{x}+1 \Rightarrow \sqrt[3]{x} = u-1 \Rightarrow x=(u-1)^3$

$\frac{dx}{du} = 3(u-1)^2 \Rightarrow dx = 3(u-1)^2\,du$

$$= \int \frac{1}{u}3(u-1)^2\,du$$

$$= 3\int \frac{u^2-2u+1}{u}\,du = 3\int\left(u-2+\frac{1}{u}\right)du = 3\left(\frac{u^2}{2}-2u+\ln|u|\right)+C$$

$$= \frac{3}{2}(\sqrt[3]{x}+1)^2 - 6(\sqrt[3]{x}+1) + 3\ln|\sqrt[3]{x}+1| + C$$

$= \ldots$

Eksempel: $\int e^{\sqrt{x}}\,dx$ $\qquad u=\sqrt{x} \Rightarrow u^2=x \Rightarrow x=u^2$, $\quad dx = 2u\,du$

$$= \int \underbrace{e^u}_{V'}\cdot\underbrace{2u}_{U}\,du$$

Delvis integrasjon: $U=2u,\ V'=e^u$; $U'=2,\ V=e^u$

$$= 2ue^u - \int 2e^u\,du$$
$$= 2ue^u - 2e^u + C$$
$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

Substitusjon i bestemte integraler

$$\int_a^b f(g(x))\,dx = \int_{g(a)}^{g(b)} f(u)h'(u)\,du$$

$u=g(x) \Rightarrow x=h(u)$

$dx = h'(u)\,du$

$x=a,\ u=g(a)$

$x=b,\ u=g(b)$

Eksempel: $\int_0^1 \arcsin x\,dx$ $\qquad u=\arcsin x \Rightarrow x=\sin u$, $\quad dx = \cos u\,du$

$$= \int_0^{\frac{\pi}{2}} \underbrace{u}_{U}\,\underbrace{\cos u}_{V'}\,du$$

Delvis: $U=u,\ V'=\cos u$; $U'=1,\ V=\sin u$

$u(0) = \arcsin(0) = 0$

$u(1) = \arcsin 1 = \frac{\pi}{2}$

$$= \left[u\sin u\right]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin u\,du$$

$$= \frac{\pi}{2}\cdot 1 - 0 - \left[-\cos u\right]_0^{\pi/2} = \frac{\pi}{2} + \left[\cos u\right]_0^{\pi/2} = \frac{\pi}{2} + \underbrace{\cos\frac{\pi}{2}}_{0} - \underbrace{\cos 0}_{1} = \frac{\pi}{2} - 1$$