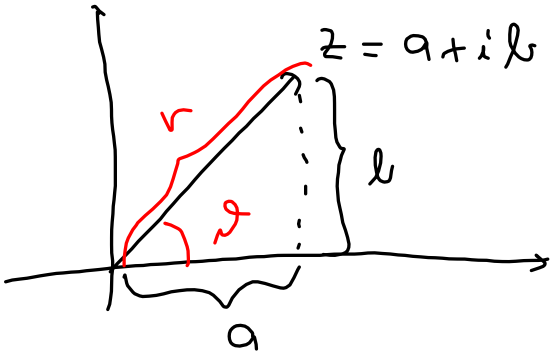
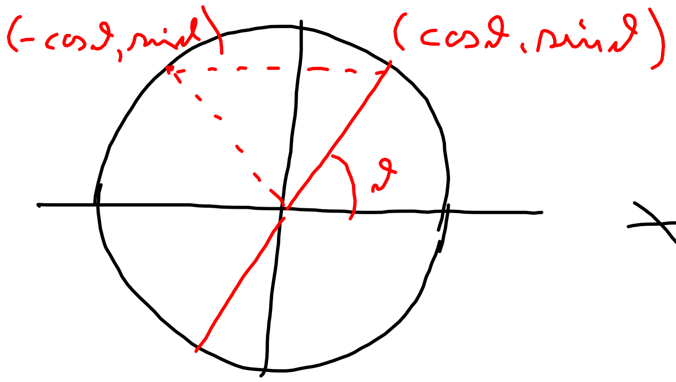
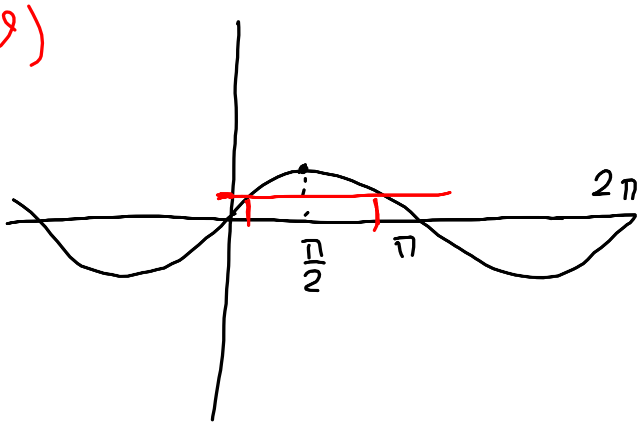
Komplekse tall



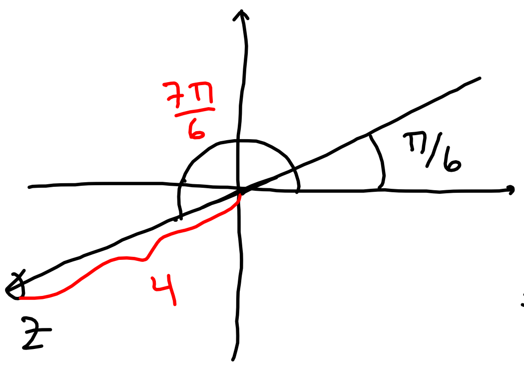
$$z = a + ib = r\cos\theta + i\, r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$1/2$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	ikke def





Eksempel: z har polarkoordinater $r = 4$ og $\theta = \frac{7\pi}{6}$. Skriv z på formen $z = a + ib$



$$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$$

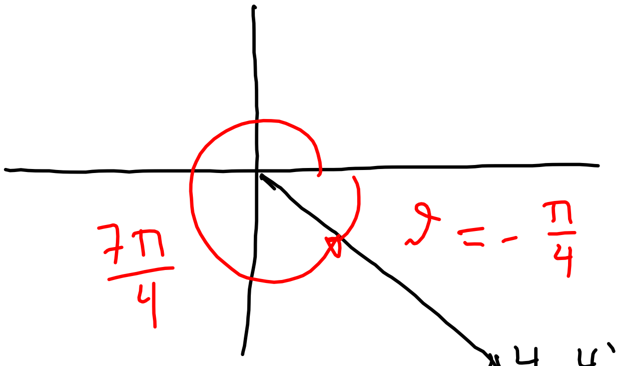
$$z = \overset{r}{4}\cos\frac{7\pi}{6} + i\,\overset{r}{4}\sin\frac{7\pi}{6}$$

$$= 4\underbrace{\left(-\cos\frac{\pi}{6}\right)}_{-\sqrt{3}/2} + i\,4\underbrace{\left(-\sin\frac{\pi}{6}\right)}_{-1/2}$$

$$= \underline{-2\sqrt{3} - 2i}$$

Eksempel: Finn polarkoordinatene til

$z = 4-4i$.



$$r = \sqrt{a^2+b^2} = \sqrt{4^2+(-4)^2} = \sqrt{2\cdot 4^2} = \underline{4\sqrt{2}}$$

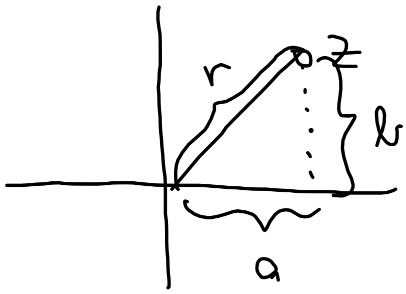
$$\sin\theta = \frac{b}{r} = \frac{-4}{4\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$\Rightarrow \theta = -\frac{\pi}{4}$ (siden vi er i 4. kvadrant)

$r = 4\sqrt{2}$, $\theta = -\frac{\pi}{4}$

$\theta = \frac{7\pi}{4}$

Geometriske tolkninger



Lengden til z = modulus til z

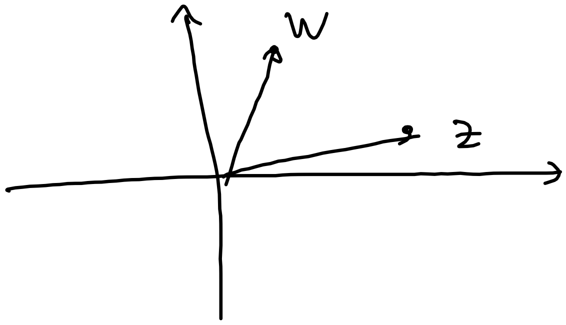
= absoluttverdien til z =

$$|z| = \sqrt{a^2+b^2}$$

Observasjon: $z\cdot\bar{z} = (a+ib)(a-ib) = a^2 - i^2 b^2 = a^2+b^2 = |z|^2$

Altså: $\boxed{|z|^2 = z\cdot\bar{z}}$ ← nyttig i oppgaver med $|z|$

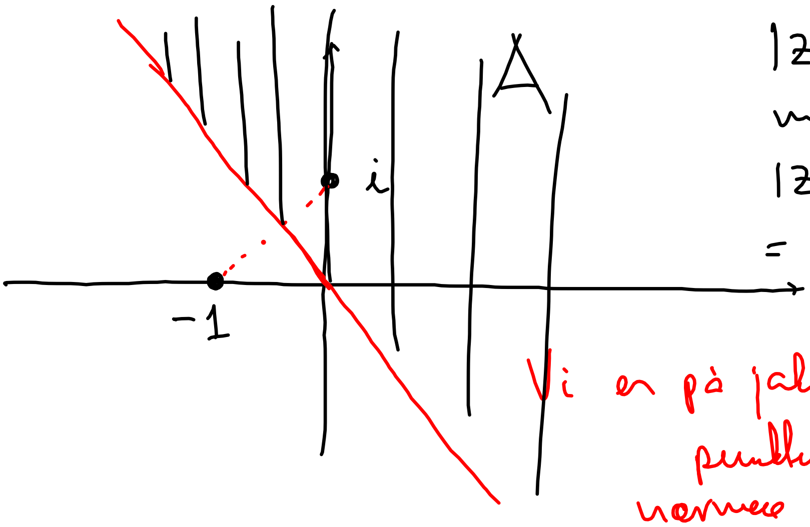
Avstanden mellom to komplekse tall:



$|z-w|$

<u>Eksempel:</u> Tegn mengden

$A = \{ z : |z-i| < |z+1| \}$ = mengden av alle z slik at $|z-i| < |z+1|$



$|z-i|$ avstanden mellom z og i

$|z+1| = |z-(-1)|$ = avstanden mellom z og -1

Vi er på jakt etter de punktene som ligger nærmere i enn -1

Eksponentialfunksjonen

Spørsmål: Hvis $z = a + ib$, hva er e^z ?

Kriterier: (i) $e^{a+i0} = e^a$

(ii) $\underbrace{e^x e^y = e^{x+y}}$ (bør også holde for komplekse tall)

{ multipliser tallene ved å addere eksponentene

Parallell

{ multipliserer komplekse tall ved å addere argumentene .....

Definisjon: Hvis $z = a + ib$ er et komplekst tall, definerer vi

$$e^z = e^{a+ib} = e^a(\cos b + i \sin b),$$

dvs et komplekst tall med modulus e^a og argument b.

Er kriteriene oppfylt?

$$e^{a+i0} = e^a(\underbrace{\cos 0}_{1} + i \underbrace{\sin 0}_{0}) = e^a \quad \text{HURRA}$$

Setning: Dersom z og w er to komplekse tall, så er

$$e^z \cdot e^w = e^{z+w}$$

Bevis: La $z = a+ib$, $w = c+id$. Da er

$$z+w = (a+c)+i(b+d).$$

Dermed er

$$e^{z+w} = e^{(a+c)+i(b+d)} =$$

$$= e^{a+c}\left(\cos(b+d)+i\sin(b+d)\right)$$

Ser så på

$$\underset{\substack{\text{modulus } e^a \\ \text{argument } b}}{e^z} \cdot \underset{\substack{\text{modulus } e^c \\ \text{argument } d}}{e^w} = \underbrace{e^z \cdot e^w}_{\substack{\text{modulus } e^a \cdot e^c \\ \text{argument } b+d}}$$

$$\boxed{\begin{aligned} e^z &= e^{a+ib} \\ &= e^a(\cos b + i \sin b) \end{aligned}}$$

$$= \underbrace{e^a \cdot e^c}_{e^{a+c}}\left(\cos(b+d)+i\sin(b+d)\right)$$

$$= \underline{e^{a+c}\left(\cos(b+d)+i\sin(b+d)\right)} = \underline{e^{z+w}}$$

Spesialtilfeller: $e^{ib} = e^{0+ib} = e^{0}(\cos b + i \sin b)$

$= \cos b + i \sin b$

Altså:

$$\boxed{e^{i\vartheta} = \cos\vartheta + i\sin\vartheta}$$

$\vartheta = \pi$: $e^{i\pi} = \underbrace{\cos\pi}_{-1} + i\underbrace{\sin\pi}_{0} = -1$

Eulers formel: $\boxed{e^{i\pi} = -1}$

$\vartheta = 2\pi$ $e^{i2\pi} = \underbrace{\cos 2\pi}_{1} + i\underbrace{\sin 2\pi}_{0} = 1$

$$\boxed{e^{2\pi i} = 1}$$

Merk: $e^{a+ib} = e^{a}\underbrace{(\cos b + i \sin b)}_{e^{ib}} = e^{a} \cdot e^{ib}$

Komplekse tall på eksponentialform

$$z = r\underbrace{(\cos\vartheta + i\sin\vartheta)}_{e^{i\vartheta}} = re^{i\vartheta}$$

$$z_1 = r_1 e^{i\vartheta_1}, \quad z_2 = r_2 e^{i\vartheta_2}$$

$$z_1 z_2 = r_1 r_2 \underbrace{e^{i\vartheta_1} \cdot e^{i\vartheta_2}} = \underbrace{r_1 r_2}\, e^{i\overbrace{(\vartheta_1+\vartheta_2)}}$$

Observasjon: $e^{z_1} \cdot e^{z_2} \cdot e^{z_3} \cdots e^{z_n} = e^{z_1+z_2+\cdots+z_n}$

Spesiell:

$$(e^z)^n = e^z \cdot e^z \cdots e^z = e^{z+z+\cdots+z} = e^{nz}$$

Dermed

$$\boxed{(e^z)^n = e^{nz}}$$

De Moivres formel: Hvis n er et naturlig tall så

$$(\cos\vartheta + i\sin\vartheta)^n = \cos n\vartheta + i\sin n\vartheta$$

Bevis: $(\cos\vartheta + i\sin\vartheta)^n = (e^{i\vartheta})^n = e^{in\vartheta}$

$$= \cos n\vartheta + i\sin n\vartheta.$$

Eksempel: Med $n=3$:

$$(\cos\vartheta + i\sin\vartheta)^3 = \underbrace{\cos 3\vartheta + i\sin 3\vartheta}$$

$$\| $$

$$\cos^3\vartheta + 3i\cos^2\vartheta\sin\vartheta - 3\cos\vartheta\sin^2\vartheta - i\sin^3\vartheta$$

$$= (\cos^3\vartheta - 3\cos\vartheta\sin^2\vartheta) + i(3\cos^2\vartheta\sin\vartheta - \sin^3\vartheta)$$

Altså $\cos 3\vartheta = \cos^3\vartheta - 3\cos\vartheta\underbrace{\sin^2\vartheta}_{(1-\cos^2\vartheta)}$

$$= \cos^3\vartheta - 3\cos\vartheta + 3\cos^3\vartheta = 4\cos^3\vartheta - 3\cos\vartheta$$

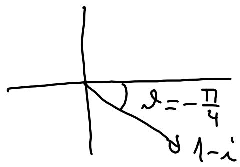
$$\sin 3\vartheta = 3\cos^2\vartheta\sin\vartheta - \sin^3\vartheta = \ldots\ldots$$

Eksempel: Hva er $(1-i)^{231}$?

Skriver $1-i$ på polarform:

$$1-i = \sqrt{2}\,(\cos(-\tfrac{\pi}{4}) + i\sin(-\tfrac{\pi}{4}))$$

$$= 2^{1/2} e^{-i\frac{\pi}{4}}$$



$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$= 2^{1/2}$

Dermed

$$(1-i)^{231} = (2^{1/2}\cdot e^{-i\frac{\pi}{4}})^{231}$$

$$= 2^{231/2}(e^{-i\frac{\pi}{4}})^{231} = 2^{231/2}\, e^{-i\frac{231}{4}\pi}$$

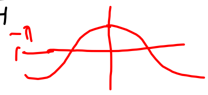
$231 : 4 = 57$ Altså $\frac{231}{4} = 57 + \frac{3}{4}$

20

31

28

3

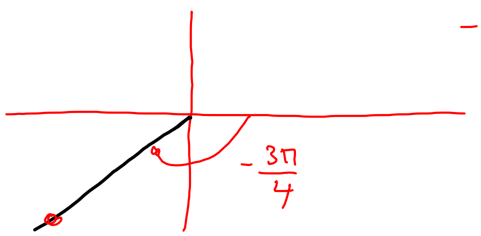


$$(1-i)^{231} = 2^{231/2}\, e^{-i(57\pi + \frac{3}{4}\pi)}$$

$e^{-i\pi} = \underbrace{\cos(-\pi)}_{-1} + \underbrace{i\sin(-\pi)}_{0}$

$$= 2^{231/2}\underbrace{(e^{-i\pi})^{57}}_{(-1)^{57} = -1}\cdot e^{-i\frac{3\pi}{4}}$$

$$= -2^{231/2}\, e^{-i\frac{3\pi}{4}} = -2^{231/2}\left[\underbrace{\cos(-\tfrac{3\pi}{4})}_{-\frac{\sqrt{2}}{2}} + i\underbrace{\sin(-\tfrac{3\pi}{4})}_{-\frac{\sqrt{2}}{2}}\right]$$



aug 21-09:42