Delbrøkoppspalting

Minner om den enkleste utgaven:

$$\int \frac{x+8}{x^2+x-6}\,dx$$

Faktoriserer nevner:

$$x^2+x-6=0$$

$$x = \frac{-1 \pm \sqrt{(-1)^2-4\cdot 1\cdot(-6)}}{2\cdot 1} = \frac{-1\pm 5}{2} = \begin{cases} 2 \\ -3 \end{cases}$$

$$x^2+x-6 = \underline{(x-2)(x+3)}$$

Altså

$$\frac{x+8}{x^2+x-6} = \frac{x+8}{(x-2)(x+3)} = \underline{\frac{A}{x-2} + \frac{B}{x+3}}$$

Må finne konstanter A og B

Ganger med $(x-2)(x+3)$:

$$x+8 = A(x+3) + B(x-2) = (A+B)x + 3A-2B$$

skal holde for alle x

$$\left.\begin{aligned} A+B&=1 \\ 3A-2B&=8 \end{aligned}\right\}$$ matcher koeffisienter

Løser

$$3A-2B=8$$
$$2A+2B=2$$
$$5A = 10 \Rightarrow A = 2$$
$$A+B=1 \Rightarrow B=-1$$

Dermed:

$$\int \frac{x+8}{x^2+x-6}\,dx = \int\left(\frac{2}{x-2} - \frac{1}{x+3}\right)dx$$

$$= \underline{\underline{2\ln|x-2| - \ln|x+3| + C}}$$

Spørsmål: 1 Hva skjer når vi ikke kan faktorisere nevner, f.eks

$$\int \frac{x+8}{x^2+2x+5}\,dx \ ?$$

2 Hva skjer med repeterte faktorer, f.eks

$$\int \frac{2x-3}{(x-1)^2(x+3)} = \frac{A}{x-1} + \cancel{\frac{B}{x-1}} + \frac{C}{x+3}$$

Terminologi: Hvis $P(x)$ og $Q(x)$ er to polynomer, kalles

$$R(x) = \frac{P(x)}{Q(x)}$$ er rasjonal funksjon.

Vi vil ha en metode for å integrere $R(x)$.

Eksempel: $\int \frac{x^3+2x^2-3x+4}{x^2-x+2}\,dx$ Her er graden til teller større enn graden til nevner.

I slike tilfeller starter vi med polynomdivisjon.

$$x^3+2x^2-3x+4 : x^2-x+2 = x+3$$

ufullstendig kvotient

$$\begin{array}{l} -(x^3-x^2+2x) \\ \hline 3x^2-5x+4 \\ -(3x^2-3x+6) \\ \hline -2x-2 \end{array}$$

rest

$$\frac{x^3+2x^2-3x+4}{x^2-x+2} = x+3+\frac{-2x-2}{x^2-x+2}$$

$$= x+3-\frac{2x+2}{x^2-x+2}$$

graden til telleren mindre enn graden til nevneren.

Ser heretter på rasjonale funksjoner der graden til teller er mindre enn graden til nevneren.

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{P(x)}{\underbrace{(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots}_{\text{første grad}}\underbrace{(x^2+a_1x+b_1)^{n_1}(x^2+a_2x+b_2)^{n_2}\cdots}_{\text{annengrad}}}\,dx$$

algebraens fundamentalteorem

Oppløsning i delbrøker:

$$\frac{P(x)}{(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x^2+a_1x+b_1)^{n_1}(x^2+a_2x+b_2)^{n_2}\cdots} =$$

$$= \frac{A_1}{x-r_1} + \frac{A_2}{(x-r_1)^2} + \cdots + \frac{A_{m_1}}{(x-r_1)^{m_1}}$$ } ledd som stammer fra faktoren $(x-r_1)^{m_1}$

$$+ \frac{B_1}{(x-r_2)} + \frac{B_2}{(x-r_2)^2} + \cdots + \frac{B_{m_2}}{(x-r_2)^{m_2}}$$ } —"— $(x-r_2)^{m_2}$

$$+ \vdots + \frac{C_1x+D_1}{(x^2+a_1x+b_1)} + \frac{C_2x+D_2}{(x^2+a_1x+b_1)^2} + \cdots + \frac{C_{n_1}x+D_{n_1}}{(x^2+a_1x+b_1)^{n_1}}$$ } —"— $(x^2+a_1x+b_1)^{n_1}$

$+ \cdots\cdots$

Eksempel: $R(x) = \dfrac{7x^7-3x^3+1}{(x+2)^3(x-3)(x^2-2x+7)^2}$

$$= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{D}{x-3} + \frac{Ex+F}{x^2-2x+7} + \frac{Gx+H}{(x^2-2x+7)^2}$$

For å finne A, B, C osv ganger vi med fellesnevner og setter koeffisientene lik hverandre. Det gir åtte ligninger med åtte ukjente.

Løser i et enklere eksempel:

Eksempel: $R(x) = \dfrac{2x+3}{(x-1)^2(x^2+2x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+2x+2}$

Ganger med $(x-1)^2(x^2+2x+2)$:

$$2x+3 = A(x-1)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)\underbrace{(x-1)^2}_{(x^2-2x+1)}$$

$$= Ax^3 + 2Ax^2 + 2Ax - Ax^2 - 2Ax - 2A + Bx^2 + 2Bx + 2B$$

$$+ Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

$$= (A+C)x^3 + (A+B-2C+D)x^2 + (2B+C-2D)x - 2A+2B+D$$

$$\underbrace{A+C=0, \quad A+B-2C+D=0, \quad 2B+C-2D=2, \quad -2A+2B+D=3}_{\text{4 ligninger med ukjente}}$$

Vet nå: $\int \frac{P(x)}{(x-r_1)^{m_1}\cdots(x^2+a_1x+b_1)^{n_1}\cdots}\,dx$

$$= \int \frac{A_1}{x-r_1}\,dx + \int \frac{A_2}{(x-r_1)^2}\,dx + \cdots + \int \frac{A_{m_1}}{(x-r_1)^{m_1}}\,dx$$

$$+ \int \frac{B_1x+C_1}{(x^2+a_1x+b_1)}\,dx + \cdots + \int \frac{B_{n_1}x+C_{n_1}}{(x^2+a_1x+b_1)^{n_1}}\,dx$$

Hvordan løser vi disse?

Vet allerede: $\int \frac{A_1}{x-r_1}\,dx = A_1 \ln|x-r_1| + C$

$m>1$: $\int \frac{A_m}{(x-r_1)^m}\,dx = A_m \int (x-r_1)^{-m}\,dx = A_m \frac{(x-r_1)^{-m+1}}{-m+1} + C$

$$= \frac{A_m}{1-m}\,\frac{1}{(x-r_1)^{m-1}} + C$$

Vi må lære å integrere

$\int \frac{Bx+C}{x^2+ax+b}\,dx$, der x^2+ax+b ikke kan faktoriseres.

Se først på et eksempel hvor $B=0$:

Eksempel: $\int \frac{3}{\underbrace{x^2+2x}_{\text{fullfører}}+4}\,dx = \int \frac{3}{\underbrace{x^2+2x+1}_{(x+1)^2}+3}\,dx$

$$= \int \frac{3}{(x+1)^2+\underbrace{3}_{\text{synd dette ikke er 1}}}\,dx = \int \frac{\cancel{3}}{\cancel{3}\left(\frac{(x+1)^2}{3}+1\right)}\,dx$$

$u = \frac{x+1}{\sqrt{3}}$, $du = \frac{dx}{\sqrt{3}}$, $dx = \sqrt{3}\,du$

$$= \int \frac{1}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}\,dx$$

$$= \int \frac{\sqrt{3}}{u^2+1}\,du = \sqrt{3}\arctan u + C = \sqrt{3}\arctan\frac{x+1}{\sqrt{3}} + C$$

Eksempel: $\int \frac{x+3}{x^2+4x+9}\,dx$

Nevner: $N(x) = x^2+4x+9$
$N'(x) = \underbrace{2x+4}_{\text{smugler denne inn i telleren.}}$

$$= \frac{1}{2}\int \frac{2x+6}{x^2+4x+9}\,dx = \frac{1}{2}\int \frac{2x+4}{x^2+4x+9}\,dx + \frac{1}{2}\int \frac{2}{x^2+4x+9}\,dx$$

$\boxed{u = x^2+4x+9 \quad du = (2x+4)\,dx}$

$$= \frac{1}{2}\int \frac{du}{u} + \int \frac{1}{x^2+4x+9}\,dx$$

$$= \frac{1}{2}\ln|u| + \int \frac{1}{x^2+4x+9}\,dx = \frac{1}{2}\ln(x^2+4x+9) + \int \frac{1}{x^2+4x+9}\,dx$$

Gjenstår $\int \frac{1}{\underbrace{x^2+4x+9}_{\underbrace{(x^2+4x+4)}_{(x+2)^2}}}\,dx = \int \frac{1}{\underbrace{x^2+4x+4}_{(x+2)^2}+5}\,dx = \int \frac{1}{(x+2)^2+5}\,dx$

synd

$$= \int \frac{1}{5\left(\frac{(x+2)^2}{5}+1\right)}\,dx = \frac{1}{5}\int \frac{1}{\left(\left(\frac{x+2}{\sqrt{5}}\right)^2+1\right)}\,dx$$

$u = \frac{x+2}{\sqrt{5}}$, $du = \frac{1}{\sqrt{5}}\,dx$, $dx = \sqrt{5}\,du$

$$= \frac{1}{5}\int \frac{\sqrt{5}}{u^2+1}\,du = \frac{\sqrt{5}}{5}\arctan u + C$$

$$= \frac{\sqrt{5}}{5}\arctan\frac{x+2}{\sqrt{5}} + C$$

Går tilbake:

$$\int \frac{x+3}{x^2+4x+9}\,dx = \frac{1}{2}\ln(x^2+4x+9) + \frac{\sqrt{5}}{5}\arctan\frac{x+2}{\sqrt{5}} + C$$

Sammenfattende eksempel:

$$\int \frac{4x^2+2x+10}{(x-2)(x^2+2x+2)}\,dx$$

$\leftarrow$ grad 2

$\leftarrow$ grad 3

polynomdivisjon ikke nödvendig

Delbrökoppspaltingen

$$\frac{4x^2+2x+10}{(x-2)(x^2+2x+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+2} \quad \Big| \, (x-2)(x^2+2x+2)$$

$$4x^2+2x+10 = Ax^2+2Ax+2A + (Bx+C)(x-2)$$

$$= Ax^2+2Ax+2A+Bx^2-2Bx+Cx-2C$$

$$= (A+B)x^2 + (2A-2B+C)x + 2A-2C$$

$A+B=4 \Rightarrow B=4-A$

$2A-2B+C=2$

$\cancel{2A-2C=10}$

$A-C=5 \Rightarrow C=A-5$

$2A-2(4-A)+A-5=2$

$2A-8+2A+A-5=2$

$5A=15 \Rightarrow A=3$

$B=1, \; C=-2$

$$\int \frac{4x^2+2x+10}{(x-2)(x^2+2x+2)}\,dx = \int \frac{3}{x-2}\,dx + \int \frac{x-2}{x^2+2x+2}\,dx$$

$$= 3\ln|x-2| + \underbrace{\int \frac{x-2}{x^2+2x+2}\,dx}_{I_2}$$

$N(x) = x^2+2x+2$

$N'(x) = 2x+2$

$$I_2 = \frac{1}{2}\int \frac{2x-4}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - \frac{1}{2}\int \frac{6}{x^2+2x+2}\,dx$$

$$= \frac{1}{2}\ln|x^2+2x+2| + C - \frac{1}{2}\int \frac{6}{\underbrace{x^2+2x+2}_{\text{fullför}}}\,dx$$