Neste uke: Oblig. innlevering
Begynner på heftet. T.L + K.N. Flervariabel analyse med lineær algebra → heftet → bok

Oppsummering av delbrøkoppspalting:

Ønsker å integrere $\frac{P(x)}{Q(x)}$ der P, Q er polynomer:

1. Hvis graden til P ikke er mindre enn graden til Q, så polynomdivider: $$\frac{P(x)}{Q(x)} = \underbrace{p(x)}_{\text{polynom}} + \frac{R(x)}{Q(x)}$$ ← n her om den grad(R) < grad(Q)

2. Faktoriser nevner: $$\frac{R(x)}{\underbrace{(x-r_1)^{m_1}\cdots}_{\text{førstegrad}}\;\underbrace{\cdots(x^2+a_1x+b_1)^{n_1}\cdots}_{\text{annengradsfaktor}}}$$

$$\int \frac{1}{x^3+8}\,dx$$
$x^3+8=0$
$x^3=-8$

3. Gjør klar til delbrøksoppspaltingen:
$$\frac{R(x)}{(x-r_1)^{m_1}\cdots(x^2+a_1x+b_1)^{n_1}\cdots} = \frac{A_1}{x-r_1} + \frac{A_2}{(x-r_1)^2} + \dots + \frac{A_{m_1}}{(x-r_1)^{m_1}} + \dots$$
$$+ \frac{B_1x+C_1}{(x^2+a_1x+b_1)} + \frac{B_2x+C_2}{(x^2+a_1x+b_1)^2} + \dots + \frac{B_{n_1}x+C_{n_1}}{(x^2+a_1x+b_1)^{n_1}} + \dots$$

4: Finn de ukjente størrelsene $A_1, A_2, \dots, B_1, C_1, \dots$: Gang med fellesnevner og sett koeffisientene lik hverandre. Løs ligningssystemet.

5: Integrer førstegradsuttrykkene: $$\int \frac{1}{x-r_1}\,dx = \ln|x-r_1|+C$$

$m>1$: $$\int \frac{1}{(x-r_1)^m} = \frac{1}{1-m}\cdot\frac{1}{(x-r_1)^{m-1}} + C$$
$$\left(\int (x-r_1)^{-m}\,dx\right.$$

6 Integrer annengradsuttrykkene:
$$\int \frac{Bx+C}{x^2+ax+b}\,dx$$

a) Smugle den deriverte av nevneren inn i teller.

b) $\int \frac{K}{x^2+ax+b}$. Fullfør kvadratet i nevner, og omform til $$\int \frac{du}{u^2+1}$$

Eksempel: $I = \int \frac{e^{3x}}{(e^x-1)(e^{2x}-e^x+1)}\,dx$

Husk: $e^{3x} = (e^x)^3$

$e^{2x} = (e^x)^2$

Prøver: $u = e^x \Rightarrow x = \ln u$

$dx = \frac{1}{u}\,du$

$$= \int \frac{u^3}{(u-1)(u^2-u+1)} \frac{1}{u}\,du$$

$$= \int \frac{u^2}{(u-1)(u^2-u+1)}\,du$$

grad 2

grad 3

HURRA: DELBRØK

Polynomdivisjon unødvendig!

Faktorisering: Kan u^2-u+1 faktoriseres?

NEI!

$u^2-u+1=0 \Rightarrow u = \underbrace{\frac{1 \pm \sqrt{(-1)^2-4\cdot 1\cdot 1}}{2\cdot 1}}_{\text{komplekse løsninger.}}$

Delbrøkoppspalting:

$$\frac{u^2}{(u-1)(u^2-u+1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2-u+1}$$

Ganger med $(u-1)(u^2-u+1)$

$$u^2 = A(u^2-u+1) + (Bu+C)(u-1)$$

$$= Au^2 - Au + A + Bu^2 - Bu + Cu - C$$

$$= (A+B)u^2 + (-A-B+C)u + (A-C)$$

legger sammen

$$\left.\begin{array}{l} A+B = 1 \\ -A-B+C = 0 \end{array}\right\} C = 1$$

$A - C = 0 \Rightarrow A = 1$

$\Rightarrow B = 0$

Altså

$$\int \frac{u^2}{(u-1)(u^2-u+1)}\,du = \int \frac{1}{u-1}\,du + \int \frac{1}{u^2-u+1}\,du$$

$$= \ln|u-1| + \underbrace{\int \frac{1}{u^2-u+1}\,du}_{I_1}$$

Mellomregning:

$$I_1 = \int \frac{1}{u^2-u+1}\,du = \int \frac{1}{\underbrace{u^2-u+\frac{1}{4}}_{(u-\frac{1}{2})^2}+\frac{3}{4}}\,du = \int \frac{1}{(u-\frac{1}{2})^2+\frac{3}{4}}\,du$$

sym!

$$= \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}(u-\frac{1}{2})^2+1\right)}\,du = \frac{4}{3}\int \frac{1}{\left(\underbrace{\frac{2(u-\frac{1}{2})}{\sqrt{3}}}_{z}\right)^2+1}\,du$$

$$= \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}}{z^2+1}\,dz = \frac{2\sqrt{3}}{3}\arctan z + C$$

$z = \frac{2u-1}{\sqrt{3}}$

$dz = \frac{2}{\sqrt{3}}\,du$

$du = \frac{\sqrt{3}}{2}\,dz$

$$= \frac{2\sqrt{3}}{3}\arctan\frac{2u-1}{\sqrt{3}} + C$$

Dette gir: $I = \ln|u-1| + \frac{2\sqrt{3}}{3}\arctan\frac{2u-1}{\sqrt{3}} + C$

$$= \ln|e^x-1| + \frac{2\sqrt{3}}{3}\arctan\frac{2e^x-1}{\sqrt{3}} + C$$

okt 23-08:35

En tur innom seksjon 9.4

$$\int \sin^n x \cos^m x \, dx$$

gjør om til $\cos x$: $\sin^2 x = 1-\cos^2 x$

Enten m eller n er et oddetall:

oddetall

$$\int \sin^3 x \cos^2 x \, dx = \int \sin x \,(\sin^2 x \cos^2 x)\, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= \int \underbrace{\sin x \,(1-\cos^2 x)\cos^2 x \, dx}_{-du}$$

$$= \int (1-u^2)u^2(-du) = \int (u^2-1)u^2 \, du = \int (u^4 - u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Både n og m like

$$\int \cos^4 x \, dx$$

$\sin^0 x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \dfrac{1+\cos 2x}{2}$

$\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \dfrac{1-\cos 2x}{2}$

$$= \int (\cos^2 x)^2 \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4}\left(x + \sin 2x + \underbrace{\int \cos^2 2x \, dx}_{I_2}\right)$$

$\cos^2 2x = \dfrac{1+\cos 4x}{2}$

Mellomregning

$$I_2 = \int \cos^2 2x \, dx = \int \frac{1+\cos 4x}{2}\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C = \frac{1}{2}x + \frac{1}{8}\sin 4x$$

Går tilbake:

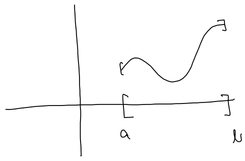
$$I = \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x\right) = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

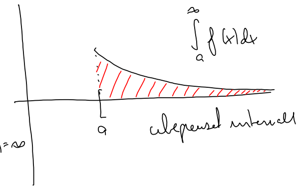
$\frac{3}{2}x$

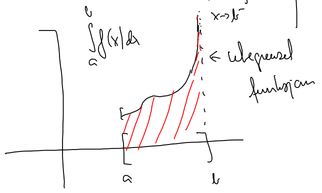
okt 23-09:14

Uegentlige integraler

Hittil: Integrert begrensede funksjoner over begrensede intervaller.







Definisjon: Anta at $f:[a,\infty)\to\mathbb{R}$ er kontinuerlig. Vi sier at $\int_a^\infty f(x)\,dx$ konvergerer dersom

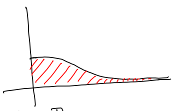
$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

eksisterer (som et tall, ikke $\pm\infty$). I så fall setter vi

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Hvis $\lim_{b\to\infty}\int_a^b f(x)\,dx$ ikke eksisterer, sier vi at integralet divergerer og det har da ingen verdi.

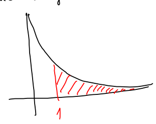
Eksempel: $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$



$= \lim_{b\to\infty}\left[\arctan x\right]_0^b = \lim_{b\to\infty}\left[\arctan b - \arctan 0\right] = \frac{\pi}{2}$

Konvergens!

Eksempel: $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$

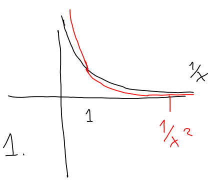


$= \lim_{b\to\infty}\left[\ln x\right]_1^b = \lim_{b\to\infty}\left[\ln b - \ln 1\right] = \infty$ Divergens!

Setning: Integralet

$$\int_1^\infty \frac{1}{x^p}\,dx$$

konvergerer for $p>1$ og divergerer for $p\leq 1$.



Bevis: Vi ser på (for $p\neq 1$):

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{b^{-p+1}}{1-p} - \underbrace{\frac{1}{-p+1}}_{\text{konstant}}\right]$$

$p<1$: $-p+1>0$, $b^{-p+1}\to\infty$

$p>1$: $-\frac{1}{-p+1} = \frac{1}{p-1}$

$-p+1<0$

$b^{-p+1}\to 0$