Eksponentialform

$$z = a + ib = \underbrace{r\cos\vartheta + i r \sin\vartheta = r e^{i\vartheta}}$$

Spesielt:

$$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$$

$$e^{-i\vartheta} = \cancel{\underbrace{\cos(-\vartheta)}_{\cos\vartheta} + i\underbrace{\sin(-\vartheta)}_{-\sin\vartheta}} = \cos\vartheta - i\sin\vartheta$$

Legger sammen

$$e^{i\vartheta} + e^{-i\vartheta} = 2\cos\vartheta \Rightarrow \cos\vartheta = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2}$$

Subtraherer:

$$e^{i\vartheta} - e^{-i\vartheta} = 2i\sin\vartheta \Rightarrow \sin\vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$$

Eksponentialfunksjon: $e^x \xrightarrow{\text{utvidet}} e^z$

for reelle x — for komplekse z

Vi utvider cosinus og sinus til komplekse tall ved å bruke formlene ovenfor

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

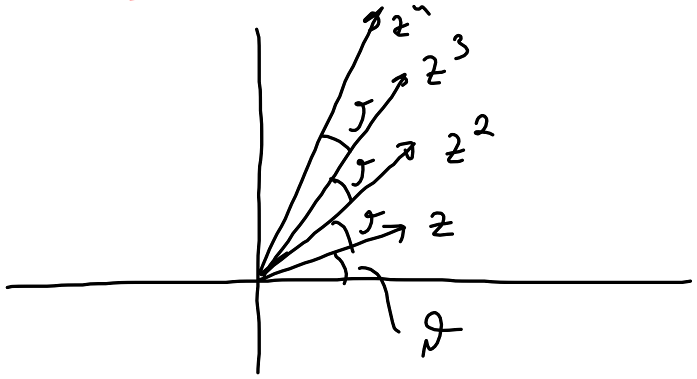
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Potenser og rötter

Vi multipliserer komplekse tall ved å multiplisere modulusene og adderer argumentene.

z modulus r, argument ϑ, $z = re^{i\vartheta}$

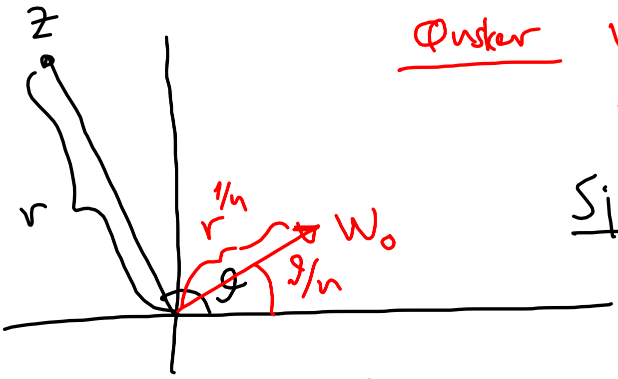
z^n modulus r^n, argument $n\vartheta$, $z^n = r^n e^{in\vartheta}$



Vi vil gå motsatt vei og finne n-te rötter:

En n-te rot av z er et komplekst tall w slik at $w^n = z$.

Anta at $z = re^{i\vartheta}$



Ønsker $w_0^n = z$

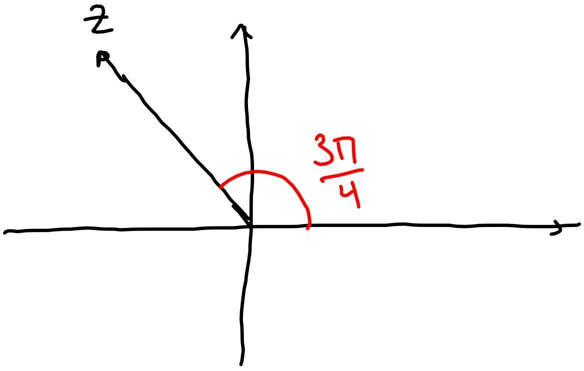
$w_0 = r^{1/n} e^{i\vartheta/n}$

Sjekk: $w_0^n = (r^{1/n} \cdot e^{i\vartheta/n})^n$

$= (r^{1/n})^n (e^{i\vartheta/n})^n = re^{i\vartheta} = z.$

Hvis $z = re^{i\vartheta}$, så er $w_0 = r^{1/n} e^{i\vartheta/n}$ en n-te rot.

Eksempel: Finn en tredjerot av $z = -4\sqrt{2} + 4\sqrt{2}i$



Trenger tallet på polarform:

$r = \sqrt{a^2 + b^2} = \sqrt{(-4\sqrt{2})^2 + (4\sqrt{2})^2}$

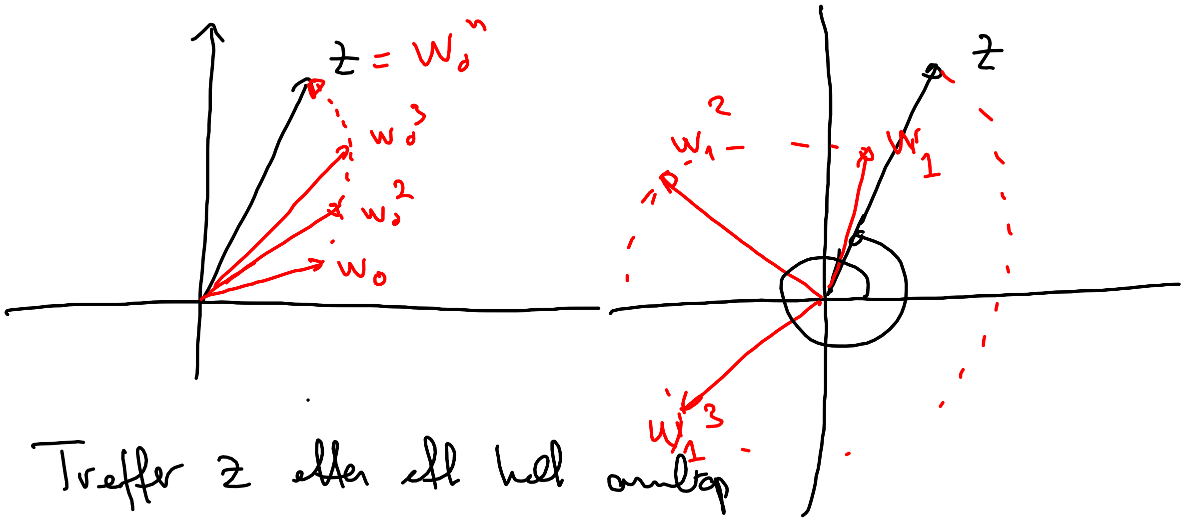
$= \sqrt{\underbrace{32 + 32}_{64}} = \underline{8}$

$\vartheta = \frac{3\pi}{4}$ dvs $z = 8e^{i\frac{3\pi}{4}}$

Følgelig er $w_0 = 8^{1/3} e^{i\frac{\pi}{4}} = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) =$

$= 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \underline{\underline{\sqrt{2} + i\sqrt{2}}}$

Finnes det flere n-te røtter?



Treffer z etter ett hel omløp

$$w_1^n = r\, e^{i(\vartheta + 2\pi)}, \text{ da blir}$$

$$w_1 = r^{1/n} e^{i\left(\frac{\vartheta + 2\pi}{n}\right)}$$

$$w_0 = r^{1/n} e^{i\vartheta/n}$$

$$w_1 = r^{1/n} e^{i\frac{\vartheta + 2\pi}{n}}$$

$$w_2 = r^{1/n} e^{i\frac{\vartheta + 4\pi}{n}}$$

$$\vdots$$

$$w_k = r^{1/n} e^{i\frac{\vartheta + 2k\pi}{n}}$$

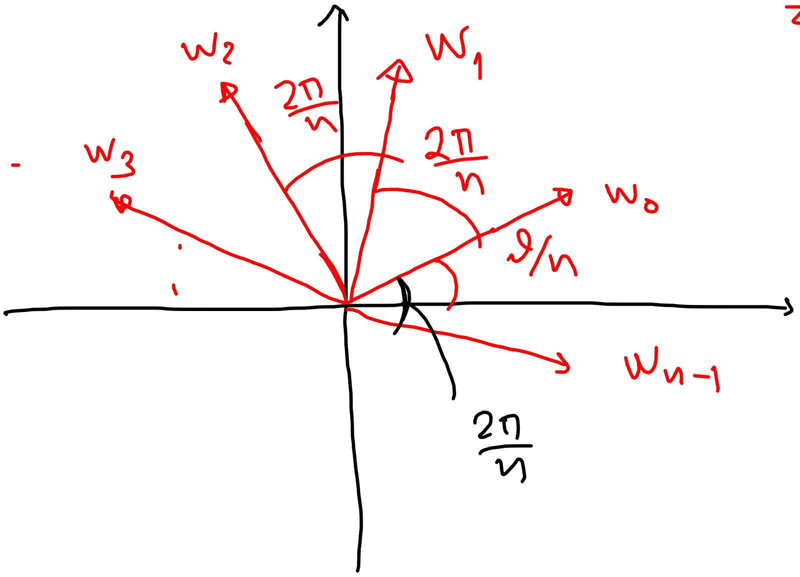
$$\vdots$$

$$w_n = r^{1/n} e^{i\frac{\vartheta + 2n\pi}{n}} = r^{1/n} e^{i\left(\frac{\vartheta}{n} + 2\pi\right)} = r^{1/n} e^{i\frac{\vartheta}{n}} = w_0$$

Konklusjon: Et komplekst tall $z \neq 0$ har mulig n n-te røtter: $w_0, w_1, \ldots, w_{n-1}$ der

$$w_k = r^{1/n} e^{i\frac{\vartheta + 2k\pi}{n}}$$

Geometrisk bilde



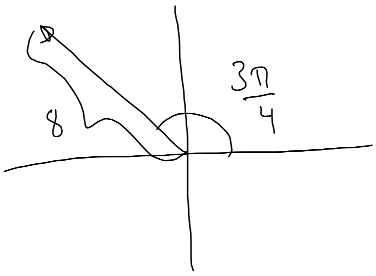
$z = re^{i\vartheta}$

$w_0 = r^{1/n} e^{i\frac{\vartheta}{n}}$

$w_1 = r^{1/n} e^{i\frac{\vartheta + 2\pi}{n}}$

Eksempel: Finn tredjerøttene til

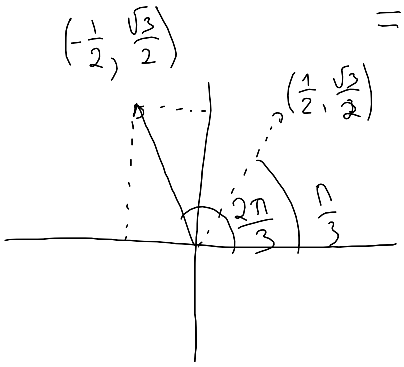
$$z = -4\sqrt{2} + 4\sqrt{2}\,i \qquad w_k = r^{1/n} e^{i\left(\frac{\theta+2k\pi}{n}\right)}$$



$$z = 8e^{i\frac{3\pi}{4}}$$

$$w_0 = 2e^{i\frac{\pi}{4}} = \sqrt{2} + i\sqrt{2}$$

$$w_1 = 8^{1/3} e^{i\left(\frac{\frac{3\pi}{4}+2\pi}{3}\right)} = 2e^{i\overbrace{\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)}^{\frac{11\pi}{12}}}$$



$$= \underbrace{2e^{i\frac{\pi}{4}}}_{w_0} \cdot e^{i\frac{2\pi}{3}} = \left(\sqrt{2} + i\sqrt{2}\right)\underbrace{\left(\cos\frac{2\pi}{3} \cancel{+ i\sin} \frac{2\pi}{3}\right)}_{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$$

$$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i - \frac{\sqrt{2}}{2}i - \frac{\sqrt{6}}{2}$$

$$= -\frac{\sqrt{6}+\sqrt{2}}{2} + \frac{\sqrt{6}-\sqrt{2}}{2}i$$

$$w_2 = 8^{1/3} e^{i\left(\frac{\frac{3\pi}{4}+4\pi}{3}\right)} = \underbrace{2e^{i\left(\frac{\frac{3\pi}{4}+2\pi}{3}\right)}}_{w_1} \cdot \underbrace{e^{i\frac{2\pi}{3}}}_{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}$$

$$= \left(-\frac{\sqrt{6}+\sqrt{2}}{2} + \frac{\sqrt{6}-\sqrt{2}}{2}i\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

= .... gjør'e sjøl!

To måter å finne røttene på:

avhengig av å kjenne disse sinusene og cosinusene

$$w_0 = r^{1/n} e^{i\theta/n}$$

$$w_1 = r^{1/n} e^{i\left(\frac{\theta+2\pi}{n}\right)} = r^{1/n}\left(\cos\left(\frac{\theta+2\pi}{n}\right) + i\sin\left(\frac{\theta+2\pi}{n}\right)\right)$$

$$= \underbrace{r^{1/n} e^{i\frac{\theta}{n}}}_{w_0} \cdot \underbrace{e^{i\frac{2\pi}{n}}}_{\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}$$

avhengig av å kjenne disse

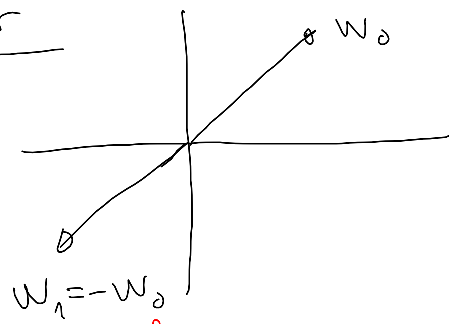
$$w_2 = w_1 e^{i\frac{2\pi}{n}}$$

$$w_3 = w_2 e^{i\frac{2\pi}{n}}$$

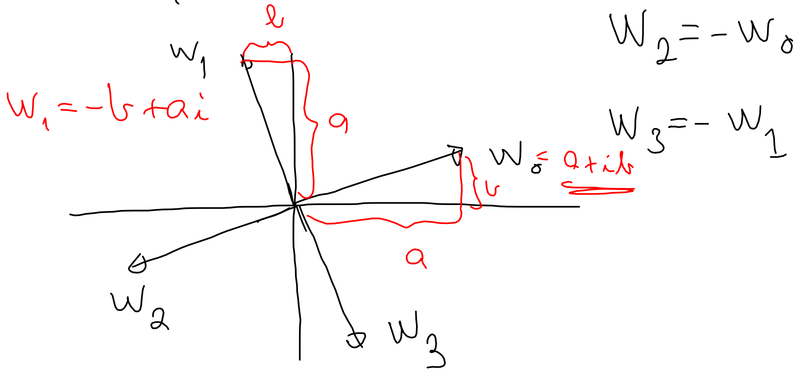
$$\vdots$$

Noen snarveier

n = 2:



n = 4



Annengradsligninger

Vet fra før: a, b, c er reelle tall, $a \neq 0$, då har ligningen $ax^2 + bx + c = 0$ løsningene

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

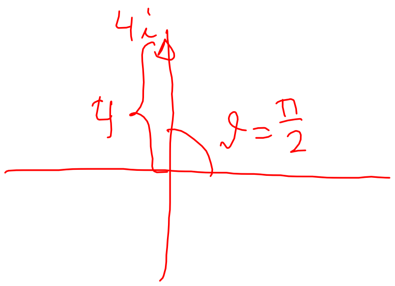
Denne formelen gjelder fortsatt når a, b, c er komplekse tall og vi tolker $\pm\sqrt{b^2 - 4ac}$ som de komplekse kvadratrøttene til $b^2 - 4ac$

Eksempel: $z^2 - 2iz - (1+i) = 0$

$$z = \frac{2i \pm \sqrt{(-2i)^2 + 4\cdot 1\cdot(1+i)}}{2\cdot 1}$$

$$= \frac{2i \pm \sqrt{-4+4+4i}}{2} = \frac{2i \pm \sqrt{4i}}{2}$$

Trenger kvadratrøttene av $4i$:



$$w_0 = 4^{1/2} e^{i\frac{\pi}{4}} = 2\left(\overbrace{\cos\frac{\pi}{4}}^{\frac{\sqrt{2}}{2}} + i\overbrace{\sin\frac{\pi}{4}}^{\frac{\sqrt{2}}{2}}\right)$$

$$= \underline{\sqrt{2} + i\sqrt{2}}$$

$$w_1 = -\sqrt{2} - i\sqrt{2}$$

$$z = \frac{2i \pm \sqrt{4i}}{2} = \frac{2i \pm (\sqrt{2} + i\sqrt{2})}{2} = \begin{cases} \frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2} i \\ -\frac{\sqrt{2}}{2} + \frac{2-\sqrt{2}}{2} i \end{cases}$$