Seksjon 3.3

10) $\sin(z+w) = \sin z \cdot \cos w + \cos z \cdot \sin w$

$z, w \in \mathbb{C}$.

Definere $\sin z$:

1) $e^{i\theta} = \cos\theta + i\sin\theta$, $\theta \in \mathbb{R}$.

2) $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$
$= \cos\theta - i\sin\theta$.

*) $\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{2\cos\theta}{2} = \cos\theta$. $\theta \in \mathbb{R}$.

e^z for $z \in \mathbb{C}$ er definert.

Setter inn z for θ:

$\Rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2}$, definert for $z \in \mathbb{C}$.

Tilsvarende:

$e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$

$= i\sin\theta + i\sin\theta = 2i\sin\theta$.

$\Rightarrow \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$, $\theta \in \mathbb{R}$.

Definerer:

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $z \in \mathbb{C}$.

$$\sin(z+w) = \sin z \cos w + \cos z \sin w.$$

$$\sin(z+w) = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i}.$$

$$HS = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\cdot\left(\frac{e^{iw}+e^{-iw}}{2}\right) + \left(\frac{e^{iz}+e^{-iz}}{2}\right)\cdot\left(\frac{e^{iw}-e^{-iw}}{2i}\right)$$

$$= \frac{(e^{iz}-e^{-iz})(e^{iw}+e^{-iw})}{4i} + \frac{(e^{iz}+e^{-iz})(e^{iw}-e^{-iw})}{4i}.$$

$$= \frac{(e^{iz}\cdot e^{iw} + e^{iz}e^{-iw} - e^{-iz}e^{iw} - e^{-iz}e^{-iw})}{4i}$$

$$+ \frac{(e^{iz}e^{iw} - e^{iz}e^{-iw} + e^{-iz}e^{iw} - e^{-iz}e^{-iw})}{4i}$$

$$= \frac{(e^{iz+iw} + e^{iz-iw} - e^{-iz+iw} - e^{-iz-iw})}{4i}$$

$$+ \frac{(e^{iz+iw} - e^{iz-iw} + e^{-iz+iw} - e^{-iz-iw})}{4i}.$$

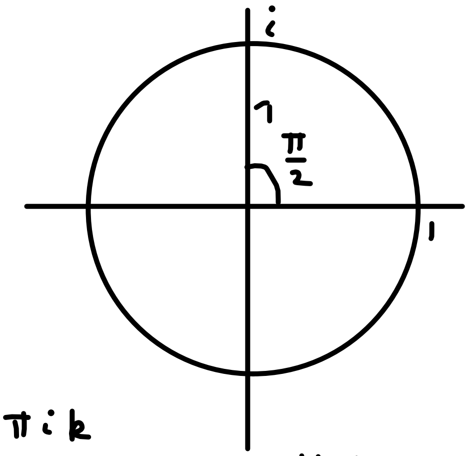
$$= \frac{1}{4i}\Big(e^{iz+iw} + e^{iz-iw} - e^{-iz+iw} - e^{-iz-iw}$$
$$+ e^{iz+iw} - e^{iz-iw} + e^{-iz+iw} - e^{-iz-iw}\Big)$$

$$= \frac{1}{4i}\left(2e^{iz+iw} - 2e^{-iz-iw}\right)$$

$$= \frac{(e^{i(z+w)} - e^{-i(z+w)})}{2i} = \sin(z+w). \quad \square.$$

Seksjon 3.4

1 Finn kvadratrøttene til z: skriv som $re^{i\theta}$ og $a+ib$.



a) $z = i$.

Modulus: $|z| = 1$

Argument: $\theta = \frac{\pi}{2}$.

$$z = re^{i\theta} = e^{i\frac{\pi}{2}} = e^{i\frac{\pi}{2} + 2\pi i k}, \text{ heltall } k.$$

Tar kvadratrot:

$$\sqrt{z} = z^{\frac{1}{2}} = \left(e^{i\frac{\pi}{2} + 2\pi i k}\right)^{\frac{1}{2}} = e^{\frac{i\frac{\pi}{2} + 2\pi i k}{2}}.$$

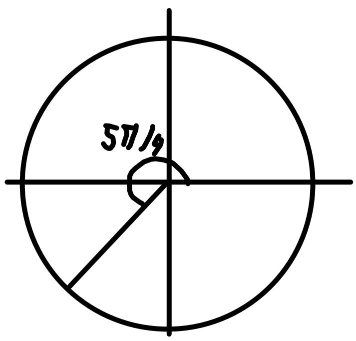
$$= e^{i\frac{\pi}{4} + \pi i k}.$$

To muligheter: $k = 0, 1$.

$$w_0 = e^{i\frac{\pi}{4} + \pi i \cdot 0} = e^{\frac{i\pi}{4}} = \underline{\underline{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.}}$$

$$w_1 = e^{\frac{i\pi}{4} + \pi i \cdot 1} = e^{\frac{i5\pi}{4}} = \underline{\underline{\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}}}$$

$$= \underline{\underline{-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}}.$$



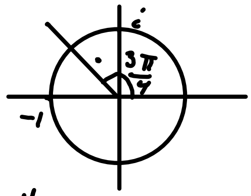
Oppgv 8 a)

Finn alle $z \in \mathbb{C}$ slik at $z^3 = -1+i$, og vis løsningsmengden på en figur.

Finne tredjerøtter til $-1+i = w$

Modulus: $r = \sqrt{(-1)^2+1^2} = \sqrt{2}$.

Argument: $\theta = \frac{3\pi}{4}$.



$$w = \sqrt{2}\, e^{i\frac{3\pi}{4} + i2\pi k} \quad k \text{ heltall.}$$

$$w^{\frac{1}{3}} = \left(\sqrt{2}\, e^{\frac{i3\pi}{4} + i2\pi k}\right)^{\frac{1}{3}} = \sqrt{2}^{\frac{1}{3}} e^{\frac{\frac{i3\pi}{4} + i2\pi k}{3}}$$

$$\left(\sqrt{2} = 2^{\frac{1}{2}} \Rightarrow \sqrt{2}^{\frac{1}{3}} = \left(2^{\frac{1}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{6}}\right)$$

$$w^{\frac{1}{3}} = 2^{\frac{1}{6}} \cdot e^{\frac{i\pi}{4} + i\frac{2\pi}{3}k}.$$

Setter inn $k = 0,1,2$.

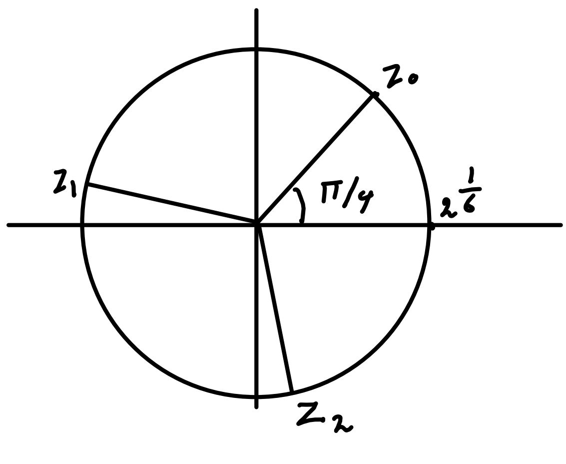
$$z_0 = 2^{\frac{1}{6}} e^{\frac{i\pi}{4}},$$

$$z_1 = 2^{\frac{1}{6}} e^{\frac{i\pi}{4} + \frac{i2\pi}{3}} = 2^{\frac{1}{6}} e^{\frac{i11\pi}{12}}.$$

$$\left(\frac{i\pi}{4} + \frac{i2\pi}{3} = \frac{i3\pi}{12} + \frac{i8\pi}{12} = \frac{i11\pi}{12}\right)$$

$$z_2 = 2^{\frac{1}{6}} e^{i\frac{\pi}{4} + i\frac{4\pi}{3}} = 2^{\frac{1}{6}} e^{i\pi + \frac{i7\pi}{12}}.$$

$$\left(\frac{i\pi}{4} + \frac{i4\pi}{3} = \frac{i3\pi}{12} + \frac{i16\pi}{12} = \frac{i19\pi}{12} = i\pi + \frac{i7\pi}{12}.\right)$$



Merk: $\frac{7\pi}{12} = \frac{\pi}{2} + \frac{\pi}{12}$

Seksjon 3.5

3 a) Finn komplekse og reelle faktoriseringer av:

·) $z^4 + 2z^2 + 1$.

Substituerer $y = z^2$. ←

·) $y^2 + 2y + 1$.

Finner nullpunkter: abc-formel gir $y = -1$.

·) $y^2 + 2y + 1 = (y+1)^2 = (z^2+1)^2$.

Ser nå på $z^2 + 1$. abc-formel gir

$$z = \frac{0 \pm \sqrt{-4}}{2} = \frac{\pm 2i}{2} = \pm i.$$

Algebraens fundamentalteorem sier at $z^2 + 1 = (z-i)(z+i)$ ←

$$\underline{z^4 + 2z^2} + 1 = (z^2+1)^2 = (z-i)^2(z+i)^2.$$

↑ reell fakt. ↑ kompleks fakt.

5 a) Vis at i er en rot i polynomet
$$P(z) = z^4 + 2z^3 + 4z^2 + 2z + 3.$$

$$P(i) = i^4 + 2i^3 + 4i^2 + 2i + 3$$
$$= 1 - \cancel{2i} - 4 + \cancel{2i} + 3$$
$$= 0.$$

b) Finn reelle og komplekse faktoriseringer av $P(z)$.

Vet at i er en rot, og $P(z)$ er et reelt polynom.
Komplekse løsninger (ikke-reelle) kommer i konjugerte par.

$\Rightarrow$ Den konjugerte av i er også en løsning. $\bar{i} = -i$.

Polynomene $(z-i)$ og $(z+i)$ er faktorer i den komplekse faktoriseringen til $P(z)$.

$\Rightarrow$ $(z-i)(z+i) = z^2+1$. er en faktor.

Polynomdividerer:

$$z^4+2z^3+4z^2+2z+3 : z^2+1 = \underline{\underline{z^2+2z+3}}$$

$$\begin{array}{l} z^4+z^2 \\ \hline 2z^3+3z^2+2z+3 \\ 2z^3+2z \\ \hline 3z^2+3 \\ 3z^2+3 \\ \hline 0 \end{array}$$

abc-formel:

$$z = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm 2\sqrt{2}\,i}{2} = \underline{\underline{-1 \pm \sqrt{2}\,i}}.$$

Løsningene er: $i, -i, -1+\sqrt{2}i, -1-\sqrt{2}i$.

Reell faktorisering:

$$P(z) = (z^2+1)(z^2+2z+3)$$

Reell fakt. fordi har ingen reelle røtter.

Kompleks faktorisering:

$$P(z) = (z-i)(z+i)(z+1-\sqrt{2}i)(z+1+\sqrt{2}i).$$

Koeffisienten er 1 fordi koeffisienten til z^4 i $P(z)$ er 1. $\square$

Seksjon 3.4

15 a) Finn løsninger til

$z^3 + iz^2 + z = 0.$

Kan faktorisere ut z:

$z(z^2 + iz + 1) = 0.$

1) $z = 0$, 2) $z^2 + iz + 1 = 0.$

abc-formel:

$$z = \frac{-i \pm \sqrt{i^2 - 4}}{2} = \frac{-i \pm \sqrt{-5}}{2}$$

Kunne ha vært komplekst. Må da bruke samme metode som i oppgv 1.

$$= \frac{-i}{2} \pm \frac{i\sqrt{5}}{2}.$$

Løsninger: $z_0 = 0$, $z_1 = -\frac{i}{2} + \frac{i\sqrt{5}}{2}$, $z_2 = -\frac{i}{2} - \frac{i\sqrt{5}}{2}$