20 spørsmål → 5 alternativer

Anverteder: Digital

- ett poeng riktig
- 0 poeng galt.

Silurveien:

$x^2 y$

$x\,2y$

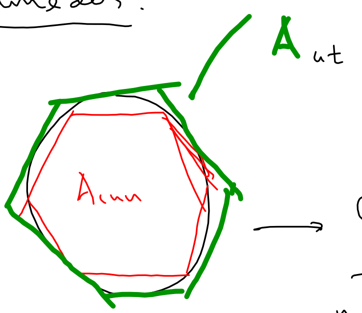
Kapittel 8

Integrasjon

Historisk:

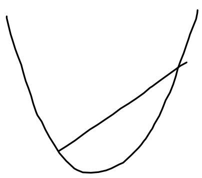
- arealberegning
- volumberegning

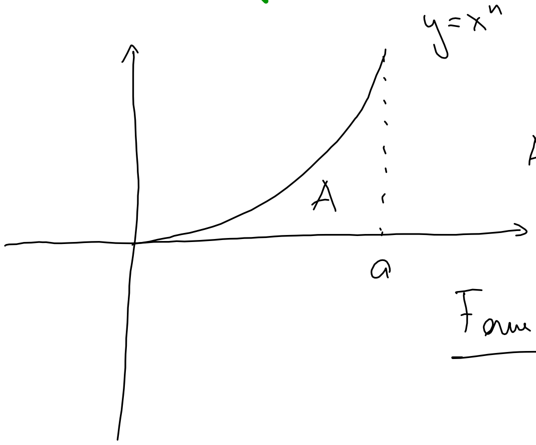
Arkimedes:

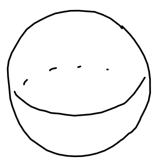


$A_{inn} \le A_{sirkel} \le A_{ut}$

→ 96-kant





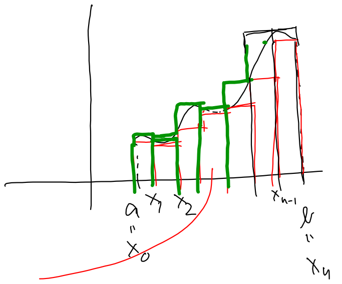


$$A = \int_0^a x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^a$$

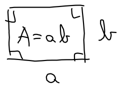
$$= \frac{a^{n+1}}{n+1}$$

Formel

Anta at $f: [a,b] \to \mathbb{R}$ er begrenset.



Hva er arealet under grafen — hvis det da finnes?



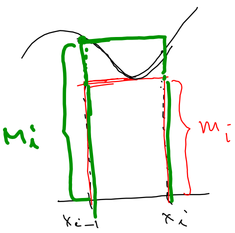
$A_{inn} < A < A_{ytre}$

En partisjon av intervallet $[a,b]$ er en mengde av punkter

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

Dette gir delintervallene:

$$[x_0,x_1],\ [x_1,x_2],\ \dots\ \underbrace{[x_{i-1},x_i]}_{i\text{-te intervall}}\ \dots\ ,[x_{n-1},x_n]$$



Høydene på boksene:

Indre bokser:

$$m_i = \inf\{f(x) : x \in [x_{i-1},x_i]\}$$

Ytre bokser:

$$M_i = \sup\{f(x) : x \in [x_{i-1},x_i]\}$$

Areal til indre boks: $m_i(x_i - x_{i-1})$

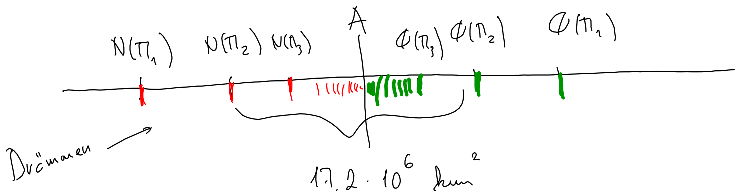
Areal til ytre boks: $M_i(x_i - x_{i-1})$

$$N(\Pi) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$ — nedre trappesum for partisjonen Π

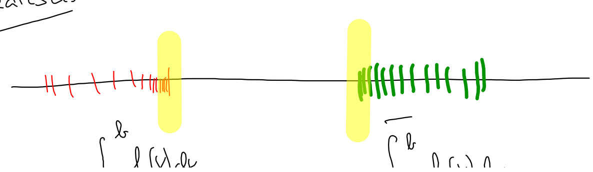
$$\text{Ø}(\Pi) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$ — øvre trappesum for partisjonen Π

Dersom området under grafen har et areal A, så

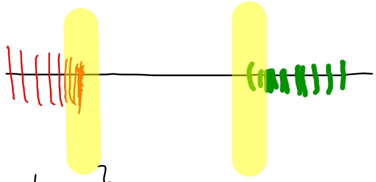
$$N(\Pi) \le A \le \text{Ø}(\Pi)$$



Mer realistisk



Øvre integral:



$$\overline{\int_a^b} f(x)\,dx = \inf\{\Phi(\Pi) : \Pi \text{ en partisjon}\}$$

Nedre integral:

$$\underline{\int_a^b} f(x)\,dx = \sup\{N(\Pi) : \Pi \text{ en partisjon}\}$$

Definisjon: Dersom $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$, så sier vi at funksjonen er integrerbar og definerer integralet $\int_a^b f(x)\,dx$ ved

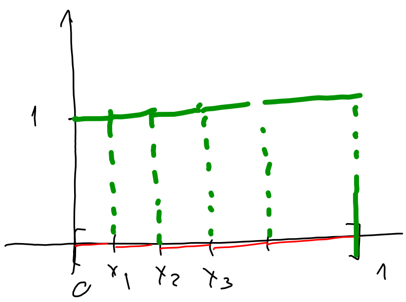
$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

Hvis ikke, så kalles funksjonen ikke-integrerbar.

Eksempel: Dirichlets funksjon

$$f(x) = \begin{cases} 1 & \text{hvis } x \text{ er et rasjonalt tall} \\ 0 & \text{hvis } x \text{ er et irrasjonalt tall} \end{cases}$$

er ikke-integrerbar.



$\Pi = \{0, x_1, x_2, \ldots, 1\}$

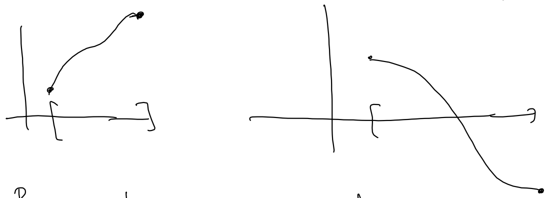
$N(\Pi) = 0$

$\Phi(\Pi) = 1$

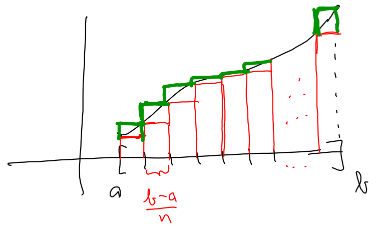
$\underline{\int_a^b} f(x)\,dx = 0$

$\overline{\int_a^b} f(x)\,dx = 1$

Setning: Alle monotone funksjoner $f: [a,b] \to \mathbb{R}$ er integrerbare



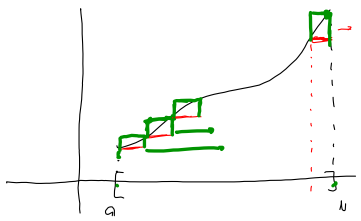
Bevis: La Π_n være partisjonen vi får ved å dele intervallet $[a,b]$ inn i n like store deler: Nok å vise at

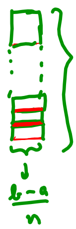


$$\lim_{n\to\infty} \left[\Phi(\Pi_n) - N(\Pi_n)\right] = 0$$

$$\Phi(\Pi_n) - N(\Pi_n) = (f(b)-f(a))\frac{b-a}{n} \to 0$$

når $n \to \infty$





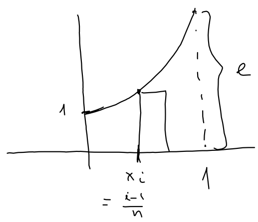
$f(b) - f(a)$

Funksjonen er integrerbar!!!!

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_{i-1})\frac{b-a}{n}}_{N(\Pi_n)}$$

$$= \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}}_{\Phi(\Pi_n)}$$

Eksempel: Hva er integralet $\int_0^1 e^x\,dx$?



$$\int_0^1 e^x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\frac{1}{n}$$

$\frac{i-1}{n}$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} e^{\frac{i-1}{n}}\cdot\frac{1}{n}$$

geometrisk rekke med $r = e^{1/n}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(e^{\frac{1}{n}}\right)^{i-1}$$

Sum av geometrisk rekke

$a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^k$

$= a_0 \frac{r^{k+1}-1}{r-1}$

$$= \lim_{n\to\infty}\frac{1}{n}\frac{\left(e^{\frac{1}{n}}\right)^n - 1}{e^{1/n}-1} = \lim_{n\to\infty}\frac{1}{n}\frac{e-1}{e^{1/n}-1}$$

$$= (e-1)\lim_{n\to\infty}\frac{\frac{1}{n}}{e^{1/n}-1} = (e-1)\lim_{n\to\infty}\frac{-\cancel{\frac{1}{n^2}}\;1}{e^{1/n}\cancel{\left(-\frac{1}{n^2}\right)}}$$

$\to 0$

$\downarrow 0$

$$= (e-1)\frac{1}{1} = \underline{e-1}$$