Substitusjon

$$\int f(g(x))\,dx = \int f(u)\,h'(u)\,du \Big|_{u=g(x)}$$

der h er den omvendte funksjonen til g.

Praksis:

$$\int f(g(x))\,dx = \int f(u)\,h'(u)\,du \Big|_{u=g(x)}$$

$u = g(x)$

$x = h(u)$

$\frac{dx}{du} = h'(u)$

$dx = h'(u)\,du$

Bestemte integraler:

$$\int_a^b f(g(x))\,dx = \int_{g(a)}^{g(b)} f(u)\,h'(u)\,du$$

$x = a$, $u = g(a)$

$x = b$, $u = g(b)$

Eksempel:

$$\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int_{\frac{\pi}{2}}^{\pi} \frac{\sin u}{\cancel{u}}\,2\cancel{u}\,du$$

$u = \sqrt{x}$ $\quad x = u^2$

$dx = 2u\,du$

$x = \frac{\pi^2}{4}$, $u = \sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}$

$x = \pi^2$, $u = \sqrt{\pi^2} = \pi$

$$= 2\int_{\frac{\pi}{2}}^{\pi} \sin u\,du = 2\Big[-\cos u\Big]_{\frac{\pi}{2}}^{\pi} = 2\Big[-\overset{-1}{\cos\pi} + \overset{0}{\cos\frac{\pi}{2}}\Big] = \underline{\underline{2}}$$

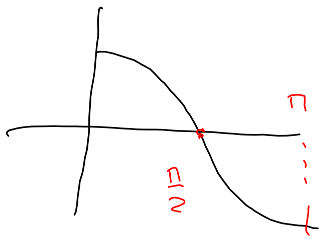
Eksempel:

$$\int e^{\arcsin x}\,dx$$

$u = \arcsin x$

$x = \sin u$

$dx = \cos u\,du$



$$= \int e^u \cos u\,du$$

Delvis integrasjon:

$U = e^u$, $V' = \cos u$

Delbrøksoppspalting

Gammelt meg...: $I = \int \frac{3x-1}{x^2+x-6}\,dx$ Faktoriser nevneren: $(x+3)(x-2) = x^2+x-6$

$$\frac{3x-1}{x^2+x-6} = \frac{3x-1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Må finne tall A og B slik at

$$\frac{3x-1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \quad \Big| \, (x+3)(x-2)$$

$$3x-1 = A(x-2) + B(x+3) = (A+B)x - 2A + 3B$$

Trenger $\left.\begin{array}{l} A+B=3 \\ -2A+3B=-1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2A+2B=6 \\ -2A+3B=-1 \end{array}\right\}$ legger sammen $5B=5$, så $B=1$

Derfor $A = 3-B = 3-1 = 2$

Altså $\frac{3x-1}{(x+3)(x-2)} = \frac{2}{x+3} + \frac{1}{x-2}$

Tilbake til integralet:

$$I = \int \frac{3x-1}{(x+3)(x-2)}\,dx = \int \left(\frac{2}{x+3} + \frac{1}{x-2}\right) dx$$

$$= 2\int \frac{1}{x+3}\,dx + \int \frac{1}{x-2}\,dx$$

$u = x+3$

$du = dx$

$$= 2\int \frac{1}{u}\,du + \ln|x-2| + C$$

$$= 2\ln|u| + \ln|x-2| + C$$

$$= \underline{\underline{2\ln|x+3| + \ln|x-2| + C}}$$

Oppsummering: $\int \frac{3x-1}{x^2+x-6}\,dx = \int \frac{3x-1}{(x+3)(x-2)}\,dx = \int \left(\frac{2}{x+3} + \frac{1}{x-2}\right)dx$

faktoriser nevneren

delbrøksoppspalting.

Hva hvis nevneren ikke kan faktoriseres?

Eksempel: $\int \frac{3}{x^2+2x+5}\,dx$

Kan ikke faktoriseres:

$x^2+2x+5=0$

$x = \frac{-2 \pm \sqrt{2^2-4\cdot 1\cdot 5}}{2} = \frac{-2\pm\sqrt{-16}}{2}$

$$= \int \frac{3}{\underbrace{x^2+2x+1}_{(x+1)^2}+4}\,dx = \int \frac{3}{(x+1)^2+4}\,dx$$

synd

$\frac{1}{(x+1)^2+1}$

$$= \int \frac{3}{4\left(\frac{(x+1)^2}{4}+1\right)}\,dx = \int \frac{3}{4\left(\left(\frac{x+1}{2}\right)^2+1\right)}\,dx$$

synd

$u = \frac{x+1}{2}$

$du = \frac{1}{2}\,dx$

$dx = 2\,du$

$$= \int \frac{3}{2\cancel{4}(u^2+1)}\,\cancel{2}\,du$$

$$= \frac{3}{2}\int \frac{1}{1+u^2}\,du = \frac{3}{2}\arctan u + C = \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C$$

Eksempel: $\int \frac{4x-2}{x^2+4x+7}\,dx$

$N(x) = x^2+4x+7$

$N'(x) = 2x+4$ ← derivert til nevneren, skal smugles inn i telleren.

$$= 2\int \frac{2x-1}{x^2+4x+7}\,dx = 2\int \frac{(2x+4)-5}{x^2+4x+7}\,dx$$

$$= 2\int \frac{2x+4}{x^2+4x+7}\,dx - \int \frac{10}{x^2+4x+7}\,dx$$ ← dritenkelt

$u = x^2+4x+7$

$du = (2x+4)\,dx$

$$= 2\int \frac{du}{u} - \int \frac{10}{x^2+4x+7}\,dx$$

$$= 2\ln|u| - \int \frac{10}{\underbrace{x^2+4x+4}_{(x+2)^2}+3}\,dx$$

$$= 2\ln(x^2+4x+7) - \int \frac{10}{(x+2)^2+3}\,dx$$

$$= 2\ln(x^2+4x+7) - \int \frac{10}{3\left(\frac{(x+2)^2}{3}+1\right)}\,dx$$

$$= 2\ln(x^2+4x+7) - \int \frac{10}{3\left(\left(\frac{x+2}{\sqrt{3}}\right)^2+1\right)}\,dx$$

$z = \frac{x+2}{\sqrt{3}}$

$dz = \frac{1}{\sqrt{3}}\,dx$

$dx = \sqrt{3}\,dz$

$$= 2\ln(x^2+4x+7) - \int \frac{10}{3(z^2+1)}\sqrt{3}\,dz$$

$$= 2\ln(x^2+4x+7) - \frac{10\sqrt{3}}{3}\int \frac{1}{1+z^2}\,dz$$

$$= 2\ln(x^2+4x+7) - \frac{10\sqrt{3}}{3}\arctan z + C$$

$$= 2\ln(x^2+4x+7) - \frac{10\sqrt{3}}{3}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + C$$

Fullføre kvadratet: $x^2+ax+b = \underbrace{x^2 + 2\cdot\frac{a}{2}x + \left(\frac{a}{2}\right)^2}_{} - \left(\frac{a}{2}\right)^2 + b$

$$\underbrace{\left(x+\frac{a}{2}\right)^2}_{\text{fullstendig}} + \underbrace{b - \left(\frac{a}{2}\right)^2}_{\text{tall}}$$

Eksempel: $x^2+6x+7 = x^2+6x+9-9+7$

Hva kan vi nå: $\int \frac{\text{førstegradspolynom}}{\text{annengradspolynom}}\,dx$

generell.

Ma $\int \frac{P(x)}{Q(x)}\,dx$ $P(x), Q(x)$ polynomer?

Faktoriseres $Q(x) = (x-r_1)(x-r_2)\dots(x^2+a_1x+b_1)(x^2+a_2x+b_2)\dots$

$$= (x-r_1)^{m_1}\dots\dots(x^2+a_1x+b_1)^{n_1}\dots$$

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{\underbrace{(x-r_1)^{m_1}}\underbrace{\dots}\underbrace{(x^2+a_1x+b_1)^{n_1}}\dots}$$

$$= \frac{A_1}{x-r_1} + \frac{A_2}{(x-r_1)^2} + \dots + \frac{A_{m_1}}{(x-r_1)^{m_1}}$$ ledd lesbed av faktoren $(x-r_1)^{m_1}$

$+ \dots$

$+ \dots$

$$+ \frac{B_1x+C_1}{x^2+a_1x+b_1} + \frac{B_2x+C_2}{(x^2+a_1x+b_1)^2} + \dots + \frac{B_{n_1}x+C_{n_1}}{(x^2+a_1x+b_1)^{n_1}}$$ ledd lesbed av faktoren $(x^2+a_1x+b_1)^{n_1}$

$+ \dots$

$+ \dots$

Eksempel: $\int \frac{7x^2+2x-5}{(x-1)(x+2)^3(x^2+2x+3)^2}\,dx$

Delbrøksoppspalting:

$$\frac{7x^2+2x-5}{(x-1)(x+2)^3(x^2+2x+3)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$
$$+ \frac{Ex+F}{x^2+2x+3} + \frac{Gx+H}{(x^2+2x+3)^2}$$

Eksempel: $\int \frac{3x^3+2x^2-4x+9}{(x^2+2x+2)(x-1)^2}\,dx$

$$\frac{3x^3+2x^2-4x+9}{(x^2+2x+2)(x-1)^2} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} \quad \Big|\,(x^2+2x+2)(x-1)^2$$

$$3x^3+2x^2-4x+9 = (Ax+B)\underbrace{(x-1)^2}_{(x^2-2x+1)} + C(x^2+2x+2)(x-1) + D(x^2+2x+2)$$

$$= Ax^3 - 2Ax^2 + Ax + Bx^2 - 2Bx + B + Cx^3 - Cx^2 + 2Cx^2 - \cancel{2Cx} + \cancel{2Cx} - 2C$$
$$+ Dx^2 + 2Dx + 2D$$

$$= (A+C)x^3 + (-2A+B+C+D)x^2 + (A-2B+2D)x + B-2C+2D$$

Må ha:
$$\left.\begin{aligned} A+C &= 3 \\ -2A+B+C+D &= 2 \\ A-2B+2D &= -4 \\ B-2C+2D &= 9 \end{aligned}\right\}$$ 4 ligninger med 4 ukjente

589