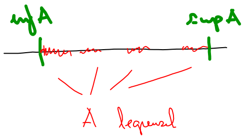
Teoretiske ryggraden

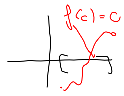
kompletthetsprinsippet



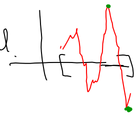
ε-δ-definisjoner

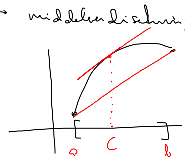
Kap 4:
- konvergens av følger
- alle begrensede, monotone konvergerer.

Kap 5:
- kontinuitet
- grenseverdier
- skjæringssetningen
- ekstremalverdisetningen



Kap 6:
- definisjon av derivert
- middelverdisetning





$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Kap 8: Integrasjon: Analysens fundamentalteorem:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for alle antideriverte F av f.

FVLA: Kap 1: – matrisemultiplikasjon, inverse matriser.

Kap 2:
- kont
- Derivasjon
  - retningsderivert
  - partiell derivert
  - gradienten
  - Jacobi matrise

Oppgavetyper

Kap 3: komplekse tall:
- $\int \frac{1}{x^3-8}\,dx$
- komplekse indreprodukt

Kap 4: ingen standardoppgaver, må kunne konvergens av følger.

Kap 5: ε-δ-oppgaver, bruk av setningene, } delt forskrift.
Kap 6: Def av derivert, middelverdisetning. }

Kap 7: Max/min, relaterte hastigheter, inverse funksjoner, cot, arcus

Kap 8: Integrasjon
- arealer
- volumer (omdreining)
- buelengder

Analysens fundamentalteorem: $\int_0^{x^3} g(x)\,dx$

Kap 9:
- skifte av variabel
- delvis integrasjon
- delbrøkoppspalting

} kombinere

- uegentlige integraler
  - regne ut
  - sammenligne med andre integraler.

FVLA:

Kapittel 1:
- matriser: multiplikasjon, inverse matriser.
- areal, volumer: determinanter, vektorprodukt.

Kapittel 2:
- partiellderiverte
- gradienten → tolkning.
- retningsderivert
- høyere ordens deriverte, Jacobi-matrise.

Kontinuasjonseksamen H16, oppgave 14

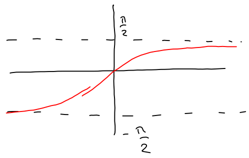
$$f(x) = \begin{cases} \arctan \frac{1}{x} & \text{når } x > 0 \\ \frac{\pi}{2} & \text{når } x = 0 \\ \pi + \arctan\frac{1}{x} & \text{når } x < 0 \end{cases}$$

a) Vis at f er kontinuerlig i 0.

Skal vise at $f(0) = \lim_{x \to 0} f(x)$. Ser på de ensidige grensene:

$$\lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} \arctan\left(\underbrace{\frac{1}{x}}_{\to\infty}\right) = \frac{\pi}{2}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} \left(\pi + \arctan\left(\underbrace{\frac{1}{x}}_{\to -\infty}\right)\right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$



Dette betyr at $\lim_{x\to 0} f(x) = \frac{\pi}{2} = f(0)$. Altså er f kontinuerlig i 0.

b) Undersøk om f er deriverbar i 0 og finn i så fall $f'(0)$.

Må undersøke om $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ eksisterer.

Ensidige grenser.

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\arctan\frac{1}{x} - \frac{\pi}{2}}{x}$$

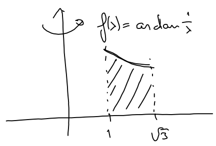
$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)}{1} = \lim_{x\to 0^+} -\frac{1}{x^2+1} = -1$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\left(\pi + \arctan\frac{1}{x}\right) - \frac{\pi}{2}}{x} \overset{L'H}{=} \lim_{x\to 0^-} \frac{\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)}{1}$$

$$= \lim_{x\to 0^+} -\frac{1}{x^2+1} = -1$$

Dette betyr at $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = -1$, som viser at f er deriverbar i 0 med $f'(0) = -1$.

c) Regn ut volumet som fremkommer når vi dreier grafen til f, $1 \le x \le \sqrt{3}$, om y-aksen.



$$V = 2\pi \int_a^b x f(x)\,dx$$

$$V = 2\pi \int_1^{\sqrt{3}} x \arctan\frac{1}{x}\,dx$$

Delvis integrasjon:
$u = \arctan\frac{1}{x}$, $v' = x$
$u' = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}$, $v = \frac{x^2}{2}$

$$I = \int x \arctan\frac{1}{x}\,dx = \frac{x^2}{2}\arctan\frac{1}{x} - \int \frac{x^2}{2}\left(-\frac{1}{x^2+1}\right)dx$$

$$= \frac{x^2}{2}\arctan\frac{1}{x} + \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx$$

polynomdivisjon
skittent triks

$$= \frac{x^2}{2}\arctan\frac{1}{x} + \frac{1}{2}\int\left[\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right]dx = \frac{x^2}{2}\arctan\frac{1}{x} + \frac{1}{2}\int\left[1 - \frac{1}{1+x^2}\right]dx$$

$$= \frac{x^2}{2}\arctan\frac{1}{x} + \frac{1}{2}x - \frac{1}{2}\arctan x + C$$

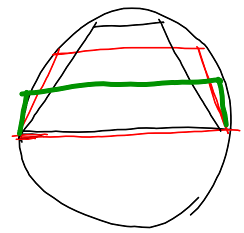
$$V = 2\pi\left[\frac{x^2}{2}\arctan\frac{1}{x} + \frac{1}{2}x - \frac{1}{2}\arctan x\right]_1^{\sqrt{3}}$$

$$= 2\pi\left[\left(\frac{3}{2}\underbrace{\arctan\left(\overset{\frac{\sqrt{3}}{3}}{\frac{1}{\sqrt{3}}}\right)}_{\frac{\pi}{6}} + \frac{\sqrt{3}}{2} - \frac{1}{2}\overset{\frac{\pi}{3}}{\arctan\sqrt{3}}\right) - \left(\cancel{\frac{1}{2}\arctan 1} + \frac{1}{2} - \cancel{\frac{1}{2}\arctan 1}\right)\right]$$

$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$, $\arctan\frac{\sqrt{3}}{3} = \frac{\pi}{6}$
$\tan\frac{\pi}{3} = \sqrt{3}$, $\arctan\sqrt{3} = \frac{\pi}{3}$

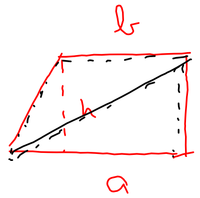
$$= 2\pi\left[\frac{\pi}{4} + \frac{\sqrt{3}}{2} - \frac{\pi}{6} - \frac{1}{2}\right] = \ldots\ldots$$

Kont. H11, nr 16:

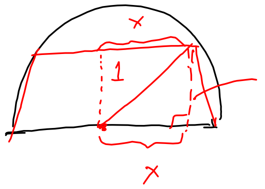


Finn trapes med størst mulig areal:

$$A = \frac{a+b}{2} \cdot h$$



$$\frac{1}{2}ah + \frac{1}{2}bh = \frac{a+b}{2} \cdot h$$



$$h = \sqrt{1^2 - x^2} = \sqrt{1-x^2}$$

$$A(x) = \frac{2+2x}{2}\sqrt{1-x^2} = (1+x)\sqrt{1-x^2}$$

$$A'(x) = 1 \cdot \sqrt{1-x^2} + (1+x)\frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = \sqrt{1-x^2} - (1+x)\frac{x}{\sqrt{1-x^2}}$$

$$0 = A'(x) \iff \sqrt{1-x^2} = \frac{(x+x^2)}{\sqrt{1-x^2}} \iff 1-x^2 = x+x^2$$

$$\iff 2x^2 + x - 1 = 0$$

abc-formel:

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{-1 \pm \sqrt{9}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$$

$$x = \frac{1}{2}$$

Eksamen 2003, oppgave 4:

$$f: (0,\infty) \to \mathbb{R}$$

(*) $f(xy) = f(x) + f(y)$

(**) $f'(1) = k$

(i) Vis at $f(1) = 0$. Setter $y = 1$ i (*):

$$f(x) = f(\underbrace{x \cdot 1}_{x}) = f(x) + f(1) \Rightarrow f(1) = 0.$$

(ii) Vis at $f(x+h) = f(x) + f\left(1 + \frac{h}{x}\right)$:

$$\underline{f(x+h)} = f\left(x\left(1 + \frac{h}{x}\right)\right) = \underline{f(x) + f\left(1 + \frac{h}{x}\right)}$$

(iii) Vis at $\underline{f'(x) = \frac{k}{x}}$ for alle x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cancel{f(x)} + f\left(1 + \frac{h}{x}\right) - \cancel{f(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} = \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right) - \underbrace{f(1)}_{= 0}}{h} = \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right) - f(1)}{\frac{h}{x}} \cdot \frac{1}{x}$$

$\to f'(1)$

$$= f'(1) \cdot \frac{1}{x} = \frac{k}{x}$$

(iv) Finn f:

$$f'(x) = \frac{k}{x}$$

$$f(x) - \underbrace{f(1)}_{0} = \int_1^x f'(t)\,dt = \int_1^x \frac{k}{t}\,dt = \left[k \ln t\right]_1^x = k \ln x$$

$$\underline{f(x) = k \ln x}$$