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<u>n-te røtter av komplekse tall</u>

Hvis z er et komplekst tall, så w er <u>n-te rot</u> av z dersom $w^n = z$.

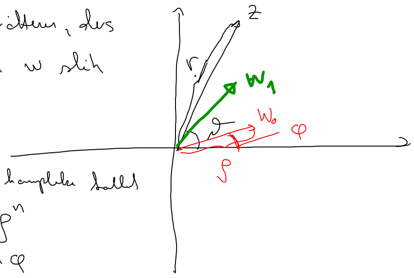
<u>Eksempler fra reelle tall</u>: 4 har to (reelle) kvadratrøtter 2 og -2

-4 har ingen reelle kvadratrøtter

8 har én tredjerot: 2 $(2^3 = 8)$

<u>Advarsel</u>: 4 har to kvadratrøtter 2 og -2, men når snakker om kvadratroten i bestemt form $(\sqrt{4})$, så mener vi alltid den positive.

Anta at z er et komplekst tall, $z \neq 0$. Vi skal finne n-te røtter, dvs komplekse tall w slik at $w^n = z$



Da er w^n det komplekse tallet med modulus ρ^n

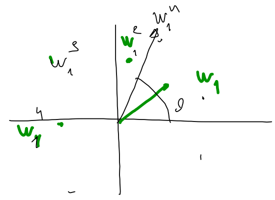
argument: $n\varphi$

Vi får $w^n = z$ ved å velge $\rho^n = r$, $n\varphi = \vartheta$

$\rho = r^{1/n}$, $\varphi = \frac{\vartheta}{n}$

Dette viser at $w_0 = r^{1/n} e^{i\frac{\vartheta}{n}}$ er en $\underline{n\text{-te rot}}$ til z.

Den neste n-te roten w_1 "treffer" z etter å ha gått én gang rundt + pluss litt til.



$w_1 = \rho e^{i\varphi}$

$\rho^n = r \Rightarrow \rho = r^{1/n}$

$n\varphi = \vartheta + 2\pi \Rightarrow \varphi = \frac{\vartheta}{n} + \frac{2\pi}{n}$

$w_1 = r^{1/n} e^{i\left(\frac{\vartheta}{n} + \frac{2\pi}{n}\right)} = r^{1/n} e^{i\frac{\vartheta + 2\pi}{n}}$

$w_2 = r^{1/n} e^{i\frac{\vartheta + 4\pi}{n}}$

$\vdots$

$w_k = r^{1/n} e^{i\frac{\vartheta + 2k\pi}{n}}$

$\vdots$

$w_{n-1} = r^{1/n} e^{i\frac{\vartheta + 2(n-1)\pi}{n}}$

$\cancel{w_n} = r^{1/n} e^{i\frac{\vartheta + 2n\pi}{n}} = \underbrace{r^{1/n} e^{i\frac{\vartheta}{n}}}_{w_0} \cdot \underbrace{e^{i2\pi}}_{1}$

Teorem: Et komplekst tall $z \neq 0$ ($= re^{i\vartheta}$) har alltid n forskjellige n-te røtter:

$w_0 = r^{1/n} e^{i\frac{\vartheta}{n}}$

$w_1 = r^{1/n} e^{i\frac{\vartheta + 2\pi}{n}}$

$w_2 = r^{1/n} e^{i\frac{\vartheta + 4\pi}{n}}$

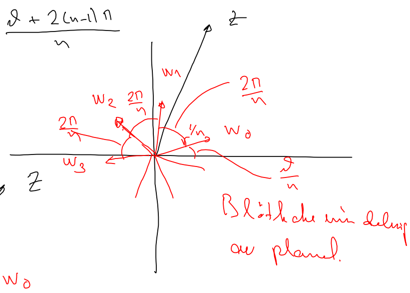
$\vdots$

$w_k = r^{1/n} e^{i\frac{\vartheta + 2k\pi}{n}}$

$\vdots$

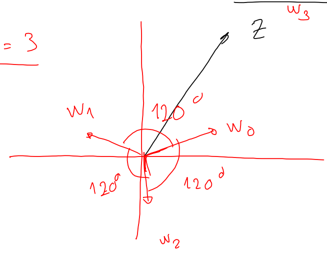
$w_{n-1} = r^{1/n} e^{i\frac{\vartheta + 2(n-1)\pi}{n}}$

Geometrisk:



Bløtkake inndeling av planet.

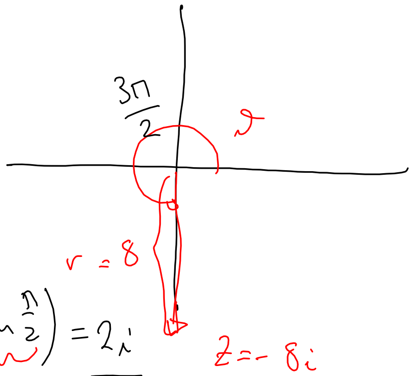
$n = 3$



Eksempel: Finn alle tredjerøtter til $z = -8i$

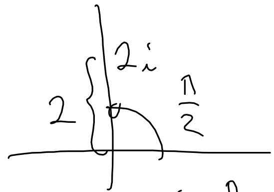
Finner polarkoordinatene til z.

$r = 8$, $\theta = \frac{3\pi}{2}$



$$w_0 = r^{1/3} e^{i\frac{\theta}{3}} = 8^{1/3} e^{i\frac{3\pi}{2}/3}$$

$$= 2e^{i\frac{\pi}{2}} = 2\left(\underbrace{\cos\frac{\pi}{2}}_{0} + i\underbrace{\sin\frac{\pi}{2}}_{1}\right) = \underline{\underline{2i}}$$

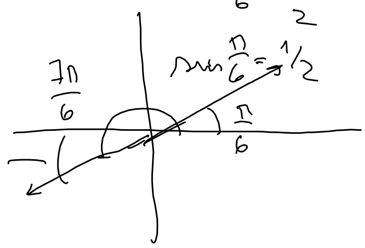


$$w_1 = r^{1/3} e^{i\frac{\theta + 2\pi}{3}} = 2e^{i\left(\frac{3\pi}{2} + 2\pi\right)/3}$$

$$= 2e^{i\left(\frac{7\pi}{2}\right)/3} = 2e^{i\frac{7\pi}{6}}$$

$$= 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$= 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \underline{\underline{-\sqrt{3} - i}}$$

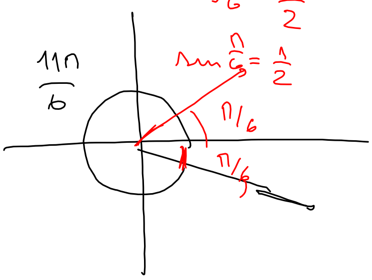


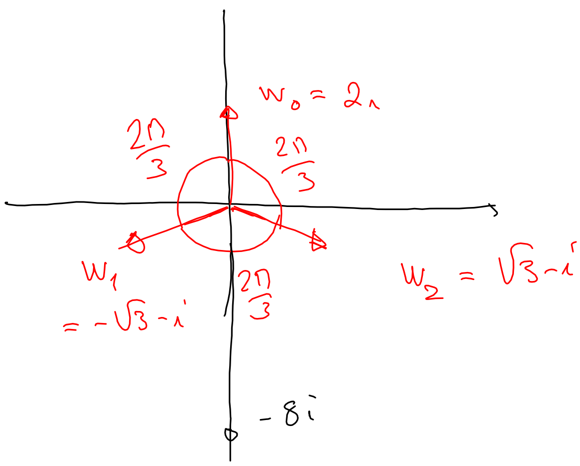
$$w_2 = r^{1/3} e^{i\frac{\theta + 4\pi}{3}}$$

$$= 2e^{i\frac{\frac{3\pi}{2} + 4\pi}{3}} = 2e^{i\frac{11\pi}{2}/3} = 2e^{i\frac{11\pi}{6}}$$

$$= 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$$= 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \underline{\underline{\sqrt{3} - i}}$$



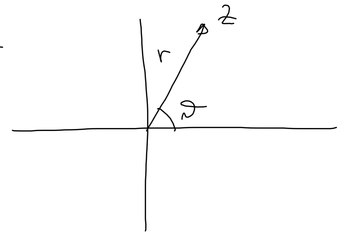


$W_0 = 2i$

$W_1 = -\sqrt{3} - i$

$W_2 = \sqrt{3} - i$

Oppsummering



$z = re^{i\vartheta}$

n-te röttene er

$w_0 = r^{1/n} e^{i\vartheta/n}$

$w_1 = r^{1/n} e^{i\frac{\vartheta+2\pi}{n}} = r^{1/n} e^{i\frac{\vartheta}{n} + i\frac{2\pi}{n}} = \underbrace{r^{1/n} e^{i\frac{\vartheta}{n}}}_{w_0} \cdot e^{i\frac{2\pi}{n}} = w_0 \underbrace{e^{i\frac{2\pi}{n}}}$

$w_2 = r^{1/n} e^{i\frac{\vartheta+4\pi}{n}}$

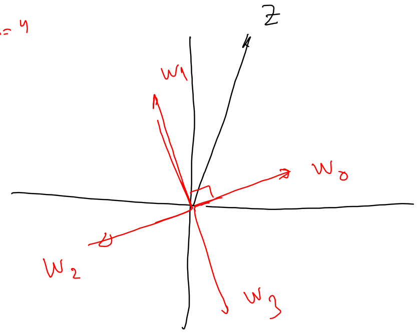
$\vdots$

$w_{n-1} = r^{1/n} e^{i\frac{\vartheta+2(n-1)\pi}{n}}$

Generelt:

$w_{k+1} = w_k e^{i\frac{2\pi}{n}}$

$n=4$

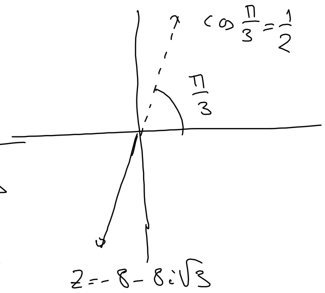


Hvis $n=4$, så $e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}}$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \underline{\underline{i}}$

Eksempel: Finn fjerderöttene til $z = -8 - 8i\sqrt{3}$

Finner polarkoordinatene:



$r = \sqrt{a^2+b^2} = \sqrt{(-8)^2 + (-8\sqrt{3})^2}$

$= \sqrt{8^2 + 8^2 \cdot 3} = 8\sqrt{1+3}$

$= \underline{\underline{16}}$

$\cos\vartheta = \frac{a}{r} = \frac{-8}{16} = -\underline{\underline{\frac{1}{2}}}$

$\vartheta = \frac{\pi}{3} + \pi = \underline{\underline{\frac{4\pi}{3}}}$

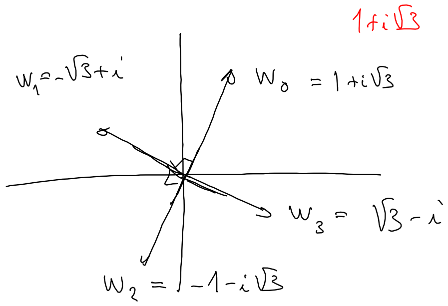
$w_0 = r^{1/4} e^{i\frac{\vartheta}{4}} = 16^{1/4} e^{i\frac{4\pi}{3}/4} = 2e^{i\frac{\pi}{3}} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$= 2(\frac{1}{2} + i\frac{\sqrt{3}}{2})$

$= \underline{1 + i\sqrt{3}}$

$w_1 = r^{1/4} e^{i\frac{\vartheta+2\pi}{4}} = \underbrace{r^{1/4} e^{i\frac{\vartheta}{4}}}_{1+i\sqrt{3}} \cdot \underbrace{e^{i\frac{\pi}{2}}}_{i} = (1+i\sqrt{3})i = i - \sqrt{3}$

$= -\sqrt{3} + i$



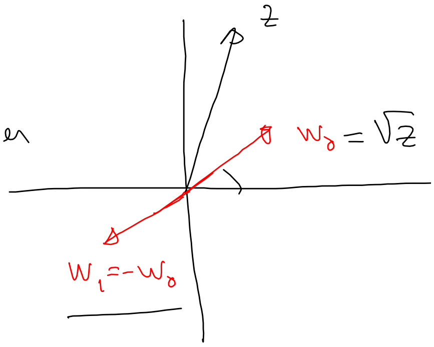
Komplekse annengradslikninger

$$ax^2+bx+c=0 \qquad a,b,c\in\mathbb{R}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \text{abc-formelen.}$$

$$az^2+bz+c=0 \qquad a,b,c\in\mathbb{C}$$

Kan bruke samme formel, må bare bli enige om hva $\sqrt{b^2-4ac}$ skal bety.



Definisjon: Med $\sqrt{z}$ der z er et komplekst tall skal vi mene den kvadratroten til z som har argument i $[0,\pi)$

Setning: Dersom a,b,c er komplekse tall, $a\neq 0$, så har ligningen

$$az^2+bz+c=0$$

løsningene

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Bevis:

$$az^2+bz+c=0 \mid 4a \qquad (2az+b)^2 = \underbrace{4a^2z^2+4azb}+b^2$$

$$\underbrace{4a^2z^2+4abz}+4ac=0$$

$$4a^2z^2+4abz+b^2-b^2+4ac=0$$

$$(2az+b)^2=b^2-4ac$$

$$2az+b=\pm\sqrt{b^2-4ac}$$

$$2az=-b\pm\sqrt{b^2-4ac}$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Eksempel: $z^2 + 2z + (1-i) = 0$

$$z = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot (1-i)}}{2\cdot 1} = \frac{-2 \pm \sqrt{4-4+4i}}{2}$$

$$= \frac{-2 \pm \sqrt{4i}}{2}$$

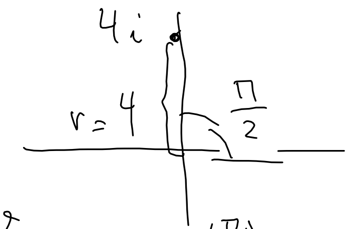
$$= \frac{-2 \pm (\sqrt{2} + i\sqrt{2})}{2}$$

$$= \begin{cases} \dfrac{-2+\sqrt{2}+i\sqrt{2}}{2} = \dfrac{-2+\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2} \\[2ex] \dfrac{-2-\sqrt{2}-i\sqrt{2}}{2} = -\dfrac{2+\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2} \end{cases}$$

Mellomregning: Hva er $\sqrt{4i}$?

$r = 4$

$\vartheta = \frac{\pi}{2}$



$$w_0 = r^{1/2} e^{i\frac{\vartheta}{2}} = 4^{1/2} e^{i(\pi/2)/2}$$

$$= 2e^{i\frac{\pi}{4}} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$

Altså $\sqrt{4i} = \sqrt{2} + i\sqrt{2}$