Delbrøksoppspalting

Eksempel: $\displaystyle\int \frac{5x^3-x^2+x}{(x-1)^2(x^2+2x+2)}\,dx$

Spalter opp integranden

$$\frac{5x^3-x^2+x}{(x-1)^2(x^2+2x+2)} = \underbrace{\frac{A}{(x-1)} + \frac{B}{(x-1)^2}}_{\text{ledd [illegible] av } (x-1)^2} + \underbrace{\frac{Cx+D}{x^2+2x+2}}_{\text{ledd [illegible] av } x^2+2x^2+2} \quad \Big|\ (x-1)^2(x^2+2x+2)$$

$$5x^3-x^2+x = A(x-1)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)(x^2-2x+1)$$

$$= Ax^3 + 2Ax^2 + \cancel{2Ax} - Ax^2 - \cancel{2Ax} - 2A + Bx^2 + 2Bx + 2B +$$

$$+ Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

$$= (A+C)x^3 + (A+B-2C+D)x^2 + (2B+C-2D)x$$

$$+ (-2A+2B+D)$$

Sammenligner koeffisienter: I: $A+C=5$

II: $A+B-2C+D=-1$

III: $2B+C-2D=1$

IV: $-2A+2B+D=0 \Rightarrow D=2A-2B$

Setter inn i ligning II:

$$-1 = A+B-2C+D = A+B-2C+2A-2B = 3A-B-2C$$

$$\Rightarrow 3A-B-2C=-1$$

Setter inn i ligning III:

$$1 = 2B+C-2D = 2B+C-4A+4B = -4A+6B+C$$

$$\Rightarrow -4A+6B+C=1$$

Setter igjen med: $A+C=5 \Rightarrow C=5-A$

$3A-B-2C=-1 \Rightarrow 3A-B-2(5-A)=-1 \Rightarrow 3A-B-10+2A=-1$

$5A-B=9$

$-4A+6B+C=1$

$\Downarrow$

$-4A+6B+5-A=1 \Rightarrow -5A+6B=-4$

Står igjen med: $\left.\begin{aligned}5A-B&=9\\-5A+6B&=-4\end{aligned}\right\}$ legger sammen $5B=5 \Rightarrow B=1$

$5A-1=9 \Rightarrow 5A=10 \Rightarrow A=2$

$C=5-A=5-2=3$

$D=2A-2B=4-2=2$

Dermed:

$$\frac{5x^3-x^2+x}{(x-1)^2(x^2+2x+2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{3x+2}{(x^2+2x+2)}$$

Integrerer:

$$\int \frac{5x^3-x^2+x}{(x-1)^2(x^2+2x+2)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx + \int \frac{3x+2}{x^2+2x+2}\,dx$$

$u=x-1$

$du=dx$

$$= 2\int \frac{1}{u}\,du + \int \frac{1}{u^2}\,du + \int \frac{3x+2}{x^2+2x+2}\,dx$$

$$= 2\ln|u| + \int u^{-2}\,du + \int \frac{3x+2}{x^2+2x+2}\,dx$$

$$= 2\ln|u| + \frac{u^{-2+1}}{-2+1} + \int \frac{3x+2}{x^2+2x+2}\,dx \qquad \frac{u^{-1}}{-1} = -\frac{1}{u}$$

$$= 2\ln|x-1| - \frac{1}{x-1} + \int \frac{3x+2}{x^2+2x+2}\,dx$$

Nevner: $N(x)=x^2+2x+2$

$N'(x)=2x+2$

Gjenstår: $I = \displaystyle\int \frac{3x+2}{x^2+2x+2}\,dx = \frac{3}{2}\int \frac{2x+\frac{4}{3}}{x^2+2x+2}\,dx$

$$= \frac{3}{2}\int \frac{2x+2-2+\frac{4}{3}}{x^2+2x+2}\,dx \qquad -2+\frac{4}{3} = -\frac{2}{3}$$

$$= \frac{3}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{1}{x^2+2x+2}\,dx$$

$v=x^2+2x+2$

$v'=2x+2$

$dv=(2x+2)\,dx$

$$= \frac{3}{2}\int \frac{dv}{v} - \int \frac{1}{(x+1)^2+1}\,dx$$

$z=x+1$

$dz=dx$

$$= \frac{3}{2}\ln|v| - \int \frac{1}{1+z^2}\,dz$$

$$= \frac{3}{2}\ln(x^2+2x+2) - \arctan z + C$$

$$= \frac{3}{2}\ln(x^2+2x+2) - \arctan(x+1) + C$$

Dermed:

$$\int \frac{5x^3-x^2+x}{(x-1)^2(x^2+2x+2)}\,dx = 2\ln|x-1| - \frac{1}{x-1} +$$

$$+ \frac{3}{2}\ln(x^2+2x+2) - \arctan(x+1) + C$$

Delbrøkoppspalting: How to do it?

$$\int \frac{P(x)}{Q(x)}\,dx$$

1. Hvis grad $(P(x)) \geq$ grad $(Q(x))$, så polynomdivider først.

Eksempel: $\int \frac{2x^2+3x-1}{x-1}\,dx$

Polynomdivisjon: $2x^2+3x-1 : x-1 = \underline{2x+5}$

$$\begin{array}{l} -(2x^2-2x) \\ \hline \quad 5x-1 \\ \quad -(5x-5) \\ \hline \quad\quad 4 \text{ - rest} \end{array}$$

$$\frac{2x^2+3x-1}{x-1} = 2x+5+\frac{4}{x-1}$$

$$\int \frac{2x^2+3x-1}{x-1}\,dx = \int \left(2x+5+\frac{4}{x-1}\right)dx = x^2+5x+4\ln|x-1|+C$$

2. Faktoriser nevneren:

$$Q(x) = (x-r_1)^{n_1} \cdots\cdots (x^2+a_1x+b_1)^{m_1} \cdots\cdots$$

3. Gjennomfør delbrøkoppspaltingen:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_1)^2} + \cdots + \frac{A_{n_1}}{(x-r_1)^{n_1}} + \cdots$$

$$+ \frac{B_1x+C_1}{(x^2+a_1x+b_1)} + \cdots + \frac{B_{m_1}x+C_{m_1}}{(x^2+a_1x+b_1)^{m_1}} + \cdots \qquad m_1 > 1$$

4. Integrer ledd for ledd:

Problem: Vet ikke hvordan vi integrerer disse!!

Løsning: kommer ikke på eksamen.

Integraler av formen $\int \sin^n x \cos^m x \, dx$ (seksjon 9.4)

To hjelpemiddler:

1. Minst én av eksponentene n, m er et oddetall:

$$I = \int \sin^5 x \cos^2 x \, dx = \int \sin x \sin^4 x \cos^2 x \, dx$$

$$\sin^2 x = 1 - \cos^2 x$$
$$\sin^4 x = (1-\cos^2 x)^2$$

$$= \int \sin x \, (1-\cos^2 x)^2 \cos^2 x \, dx$$

$$u = \cos x$$
$$du = -\sin x \, dx$$

$$= -\int (1-u^2)^2 u^2 \, du$$

$$= -\int (1-2u^2+u^4) u^2 \, du$$

$$= -\int (u^2 - 2u^4 + u^6) \, du = \text{Gjør'e sjøl!}$$

2. Både m og n er partall: $\int \sin^m x \cos^n x \, dx$

$$\cos 2x = 2\cos^2 x - 1$$
$$\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$$
$$\cos 2x = 1 - 2\sin^2 x$$
$$\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$$

Eksempel: $\int \sin^0 x \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4}\left[x + \sin 2x + \int \cos^2 2x \, dx\right]$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{4}\int \frac{1+\cos 4x}{2} \, dx$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{1}{32}\sin 4x + C = \frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

Eksempel: $\int \frac{1}{\sin x} \, dx = \int \frac{\sin x}{\sin^2 x} \, dx = \int \frac{\sin x}{1-\cos^2 x} \, dx$

$$= \int \frac{-du}{1-u^2} = \int \frac{du}{u^2-1}$$

$$u = \cos x$$
$$du = -\sin x \, dx$$

Delbrøksoppspalting: $\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1} \quad \Big| \, (u-1)(u+1)$

$$1 = A(u+1) + B(u-1) = (A+B)u + (A-B)$$

Altså: $\left.\begin{matrix} A+B=0 \\ A-B=1 \end{matrix}\right\}$ legger sammen $2A = 1 \Rightarrow A = \frac{1}{2}$

$$B = -A = -\frac{1}{2}$$

Dermed: $\int \frac{1}{u^2-1} \, du = \int \left[\frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\right] du = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$

Derfor: $\int \frac{1}{\sin x} \, dx = \frac{1}{2}\ln|\cos x - 1| - \frac{1}{2}\ln|\cos x + 1| + C$

Neste gang: Uegentlig integral: $\int_a^\infty f(x) \, dx = \lim_{b\to\infty} \int_a^b f(x) \, dx$