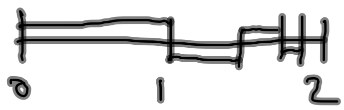
# Lindstrøms Kalkulus XII: Rekker

Eksempler

$$0{,}9999\ldots = 1$$

$$\| $$

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10{,}000} + \ldots$$



$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = 2$$

geometrisk rekke

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots = +\infty$$

divergent rekke

harmonisk rekke

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} + \ldots = \frac{\pi^2}{6}$$

L. Euler 1735

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots + \frac{1}{n^3} + \ldots$$

konvergerer

Apéry 1978

irrasjonalt tall

Geometrisk rekke $r \in \mathbb{R}$

$$1 + r + r^2 + \dots + r^n = \begin{cases} \frac{1-r^{n+1}}{1-r} & \text{hvis } r \neq 1 \\ n+1 & \text{hvis } r = 1 \end{cases}$$

Bevis ved induksjon for $n \geq 0$

$\underline{n=0}$ $1 = \frac{1-r^1}{1-r}$ ✓

$n \geq 1$.

$$1 + r + \dots + r^{n-1} + r^n = \frac{1-r^n}{1-r} + r^n = \frac{1-r^n-(1-r)r^n}{1-r} = \frac{1-r^{n+1}}{1-r} \quad \square.$$

$$\lim_{n\to\infty} (1 + r + \dots + r^n) = \begin{cases} \frac{1}{1-r} & \text{hvis } |r| < 1 \\ \text{divergerer} & \text{hvis } |r| \geq 1. \end{cases}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} r^k = \sum_{k=0}^{\infty} r^k$$

$$1 + r + r^2 + \dots + r^n + \dots = \frac{1}{1-r} \quad \text{for } |r| < 1$$

$r = -x^2$

$$1 - x^2 + x^4 - \dots + (-x^2)^n + \dots = \frac{1}{1+x^2} \quad \text{for } |x| < 1.$$

$$\int_0^x 1 - x^2 + \dots + (-x^2)^n \, dx = \int_0^x \frac{1-(-x^2)^{n+1}}{1+x^2} \, dx$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \pm \frac{x^{2n+1}}{2n+1} = \dots \quad \text{Kap 11}$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \int_0^x \frac{dx}{1+x^2} \quad |x| \leq 1$$

$= \arctan x$

Nk. Abel 12.6.9

$x = 1$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + \frac{(-1)^n}{2n+1} + \dots = \arctan 1 = \frac{\pi}{4}$$

$$\int_0^x \frac{dx}{\sqrt{x^3 - x - 1}}$$

La $\{a_n\}_{n=0}^{\infty}$ være en følge i $\mathbb{R}$

$(a_0, a_1, a_2, \ldots)$ sequence

Danner summene

$$s_0 = a_0$$
$$s_1 = a_0 + a_1$$
$$s_2 = a_0 + a_1 + a_2$$
$$\vdots$$
$$s_k = a_0 + a_1 + \ldots + a_k = \sum_{n=0}^{k} a_n$$

series

Følgen $\{s_k\}_{k=0}^{\infty}$ av summer kalles en rekke

Hva betyr $\displaystyle\sum_{n=0}^{\infty} a_n$ ? $\displaystyle\lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=0}^{k} a_n$

Def Hvis $\displaystyle s_k = \sum_{n=0}^{k} a_n \longrightarrow A$ når $k \to \infty$

sier vi at rekken $\displaystyle\sum_{n=0}^{\infty} a_n$ konvergerer

og skriver $\displaystyle\sum_{n=0}^{\infty} a_n = A.$

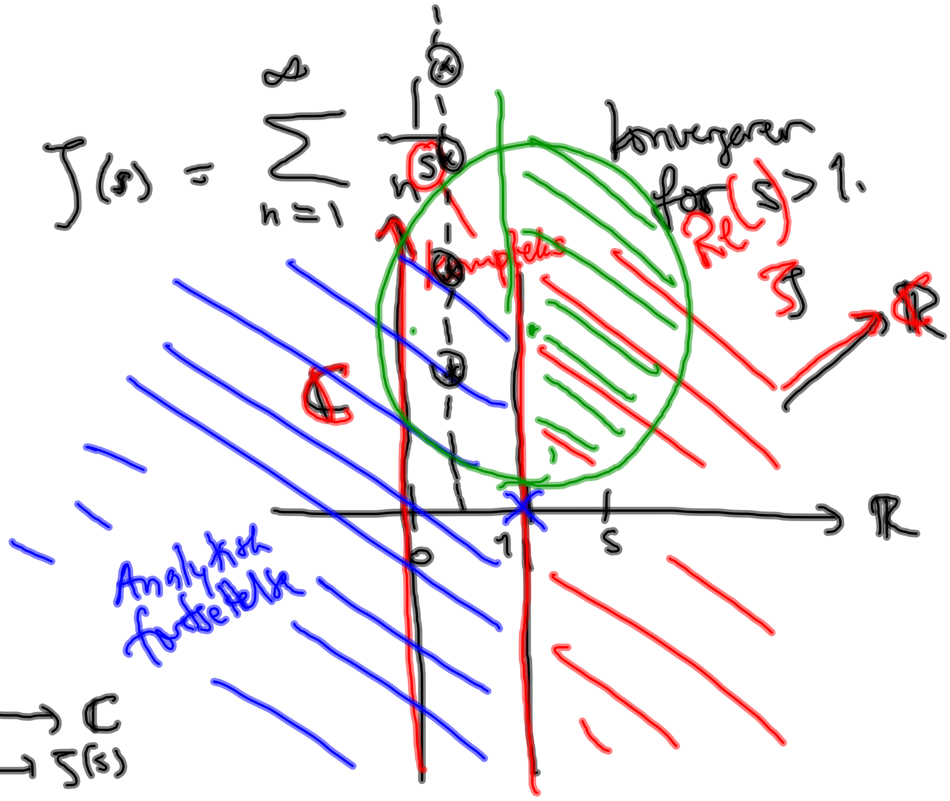
Ellers sier vi at rekken divergerer

## Eksempel

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

konvergerer for $Re(s) > 1$.

Riemann




$$\zeta : \mathbb{C} \setminus \{1\} \longrightarrow \mathbb{C}$$
$$s \longmapsto \zeta(s)$$

## Riemann-hypotesen

Hvis $\zeta(s) = 0$ og $Re(s) \in (0,1)$

er $Re(s) = \frac{1}{2}$.

<u>Setning</u> Hvis $\sum_{n=0}^{\infty} a_n = A$ konvergerer og $c \in \mathbb{R}$ vil $\sum_{n=0}^{\infty} c a_n = cA$ konvergere.

Hvis også $\sum_{n=0}^{\infty} b_n = B$ konvergerer vil

$$\sum_{n=0}^{\infty} (a_n + b_n) = A + B$$ konvergere.

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<u>Lemma</u> Hvis $\sum_{n=0}^{\infty} a_n = A$ konvergerer

så $a_n \longrightarrow 0$ når $n \longrightarrow \infty$.

<u>Bevis</u> $s_k = \sum_{n=0}^{k} a_n \longrightarrow A$ når $k \to \infty$

$s_{k-1} = \sum_{n=0}^{k-1} a_n \longrightarrow A$ når $k \to \infty$.

$a_k = s_k - s_{k-1} \longrightarrow A - A = 0$ når $k \to \infty$

$\square$

Merk Omvendingen er ikke generelt riktig:

Kan ha $a_n \to 0$ uten at $\sum_{n=0}^{\infty} a_n$ konvergerer.

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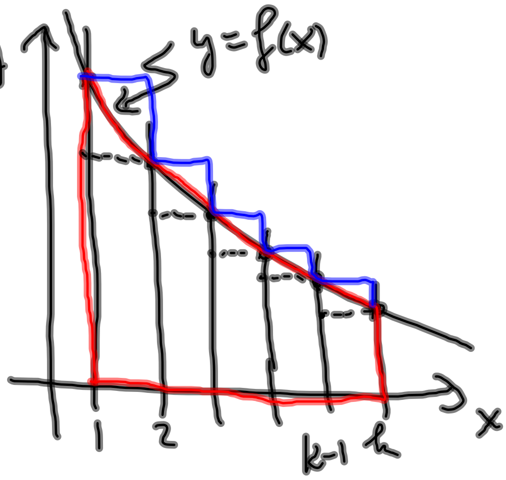
Sats $\sum_{n=1}^{\infty} \frac{1}{n}$ divergerer.

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right) + \dots$$

$$\geq 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \dots$$ som vokser over alle grenser.

## Integraltesten

La $f : [1, \infty) \to \mathbb{R}$ være en positiv, kontinuerlig og avtagende. Da er




$$\sum_{n=2}^{k} f(n) \leq \int_1^k f(x)\,dx \leq \sum_{n=1}^{k-1} f(n)$$

så rekken $\sum_{n=1}^{\infty} f(n)$ konvergerer hvis og bare hvis $\int_1^{\infty} f(x)\,dx$ konvergerer.

<u>Eks</u> $f(x) = \frac{1}{x^s} = x^{-s}$ der $s > 0$.

$$\int_1^k f(x)\,dx = \int_1^k x^{-s}\,dx = \left[\frac{1}{1-s} x^{1-s}\right]_1^k \quad s \neq 1$$

$$= \frac{1}{1-s} k^{1-s} - \frac{1}{1-s}$$

$$\longrightarrow \begin{cases} \text{divergerer} & 1-s > 0 \\ \frac{1}{s-1} & 1-s < 0, \ \Updownarrow \ s > 1 \end{cases}$$

<u>Setning</u>

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ konvergerer hvis og bare hvis $s > 1$.

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Integraltesten for $s = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div.} \iff \int_1^{\infty} \frac{1}{x}\,dx \text{ div.}$$

$$= \lim_{k \to \infty} \left[\ln x\right]_1^k$$

$$= \lim_{k \to \infty} \ln k = +\infty.$$

## Forholdstesten

La $\sum_{n=0}^{\infty} a_n$ være en rekke og anta

at $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = b$ eksisterer.

Dersom $b<1$ er rekken konvergent.

Dersom $b>1$ er rekken divergent.

Dersom $b=1$ gir testen ingen konklusjon.

Anta $a_n > 0$. Hvis $b<1$ finnes $r$ med $b<r<1$ og en $N$ slik at

$$\frac{a_{n+1}}{a_n} < r \quad \text{for alle } n \geq N.$$

$$a_{N+1} \leq r a_N, \quad a_{N+2} \leq r a_{N+1} \leq r^2 a_N$$

$$a_{N+i} \leq r^i a_N \quad \text{for } i \geq 0.$$

$$\sum_{n=0}^{\infty} a_n \text{ konvergerer} \iff \sum_{n=N}^{\infty} a_n \text{ konvergerer}$$

$$\sum_{n=N}^{N+k} a_n = \sum_{i=0}^{k} a_{N+i} \leq \sum_{i=0}^{k} r^i a_N \longrightarrow \frac{a_N}{1-r} \text{ når } k\to\infty$$

$\downarrow$

konvergerer når $k\to\infty$

Eks (MAT1110, 2009)

$x \in \mathbb{R}$

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n^2+n}}$$ for hvilke $x$ konvergerer rekken?

$$= \underbrace{\frac{(x-2)}{\sqrt{2}}}_{a_1(x)} + \underbrace{\frac{(x-2)^2}{\sqrt{6}}}_{a_2(x)} + \underbrace{\frac{(x-2)^3}{\sqrt{12}}}_{a_3(x)} + \dots$$

Forholdene er

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \left|\frac{(x-2)^{n+1}}{\sqrt{n^2+3n+2}} \frac{\sqrt{n^2+n}}{(x-2)^n}\right|$$

$$= |x-2| \sqrt{\frac{n^2+n}{n^2+3n+2}} \to |x-2| = b$$ når $n \to \infty$.

$$\left( \frac{n^2+n}{n^2+3n+2} = \frac{1+1/n}{1+3/n+2/n^2} \to \frac{1+0}{1+0+0} = 1 \text{ når } n \to \infty \right)$$

$\therefore$ Hvis $1 < x < 3 \iff |x-2| < 1$ konvergerer rekken.

Hvis $x < 1$ eller $x > 3 \iff |x-2| > 1$ divergerer rekken.

For $x = 3$ er rekken

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$$

$$\geq$$

$$\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$$

divergent

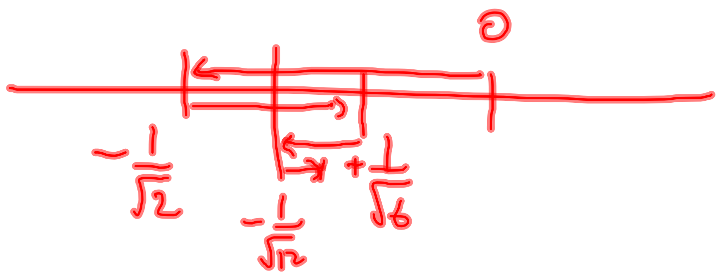
$$\frac{1}{\sqrt{n^2+n}} \overset{?\checkmark}{\geq} \frac{1}{\sqrt{2}\, n}$$

$$n^2+n \overset{\checkmark}{\leq} 2n^2$$

For $x = 1$ er rekken

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+n}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{12}} + - + \ldots$$

alternerende med ledd med absoluttverdi som avtar mot null $\Rightarrow$ konvergent




$\therefore$ Konvergensområdet er $[1,3)$.