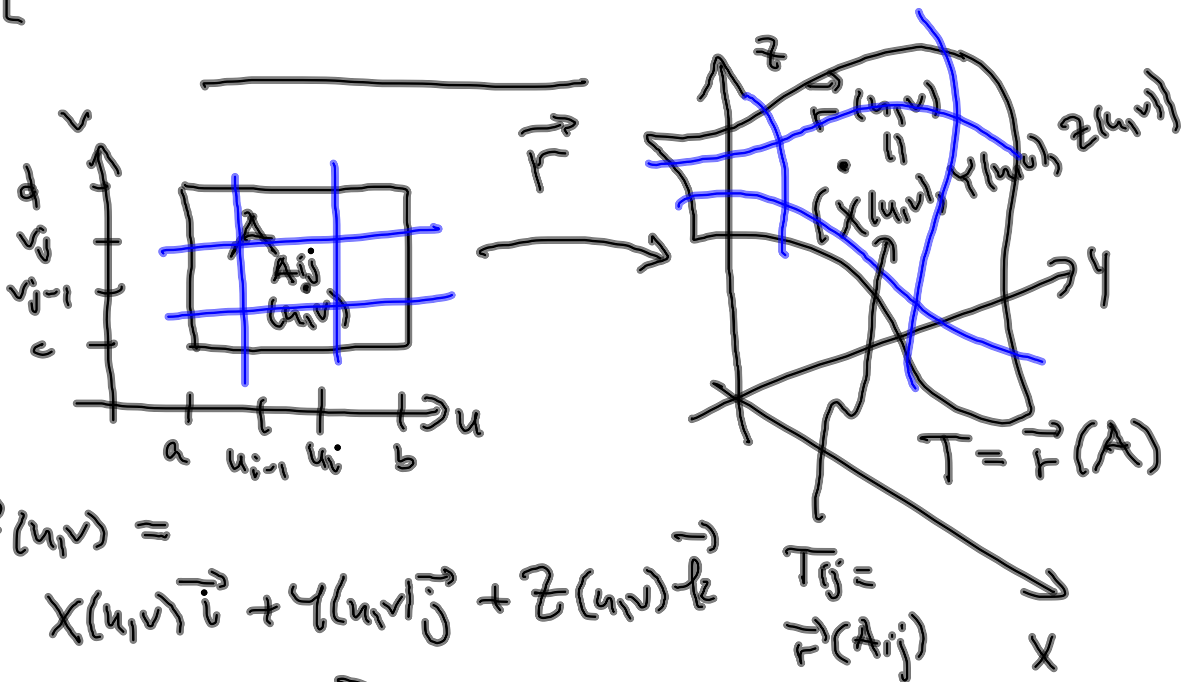
6.4 Anvendelser av dobbeltintegraler

→ parametriserte flater i $\mathbb{R}^3$

* areal
* flateintegral av skalarfelt

[– ——— " — vektorfelt]

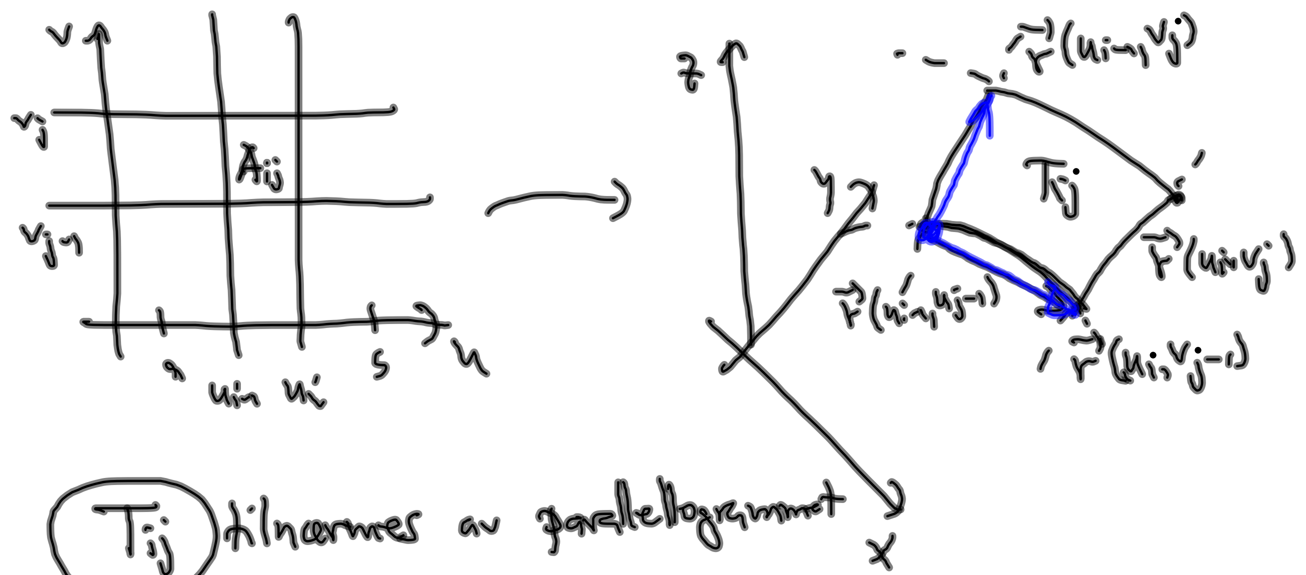


$\vec{r}(u,v) =$

$x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

$\text{areal}(T) = ?$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \text{areal}(T_{ij})$$

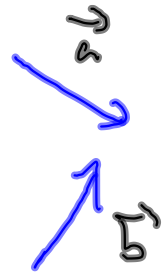


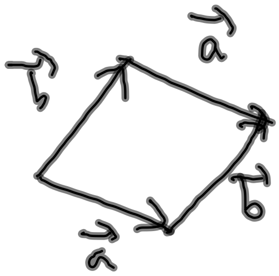
T_{ij} tilnærmes av parallellogrammet med ett hjørne i $\vec r(u_{i-1}, v_{j-1})$

og sidekanter

$\vec a = \vec r(u_i, v_{j-1}) - \vec r(u_{i-1}, v_{j-1})$

og $\vec b = \vec r(u_{i-1}, v_j) - \vec r(u_{i-1}, v_{j-1})$





med areal

$= |\vec a \times \vec b|$

Her er

$$\vec{a} = \vec{r}(u_i, v_{j-1}) - \vec{r}(u_{i-1}, v_{j-1})$$

$$\approx \frac{\partial \vec{r}}{\partial u}(u_{i-1}, v_{j-1}) \; (u_i - u_{i-1})$$

vektor skalar

og

$$\vec{b} \approx \frac{\partial \vec{r}}{\partial v}(u_{i-1}, v_{j-1}) \; (v_j - v_{j-1})$$

så

$$|\vec{a} \times \vec{b}| = \left| \frac{\partial \vec{r}}{\partial u}(u_{i-1}, v_{j-1}) \times \frac{\partial \vec{r}}{\partial v}(u_{i-1}, v_{j-1}) \right| \cdot (u_i - u_{i-1})(v_j - v_{j-1})$$

$\approx$ areal(T_{ij})

$$\text{areal}(T) \approx \sum_{i=1}^{n} \sum_{j=1}^{m} \text{areal}(T_{ij})$$

$$\approx \sum_{i=1}^{n} \sum_{j=1}^{m} \left| \frac{\partial \vec{r}}{\partial u}(\ldots) \times \frac{\partial \vec{r}}{\partial v}(\ldots) \right| (u_i - u_{i-1})(v_j - v_{j-1})$$

er en Riemann-sum for

$$(u,v) \longmapsto \left| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right|$$

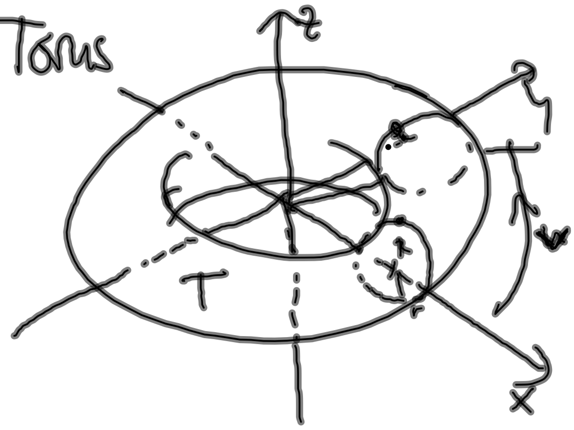
så

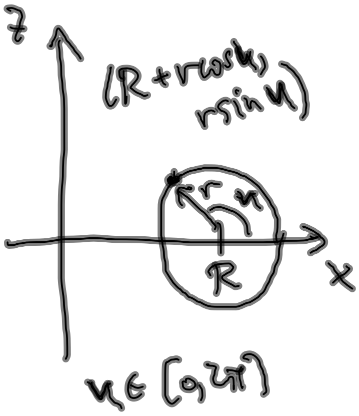
$$\text{areal}(T) = \iint_A \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| (u,v) \; du \, dv$$

$T = \vec{r}(A)$

$A = [a,b] \times [c,d]$

Eks Torus





$$\vec{r}(u,v) = (R + r\cos u)\cos v\,\vec{i} + (R + r\cos u)\sin v\,\vec{j} + r\sin u\,\vec{k}$$

$$(u,v) \in A = [0,2\pi] \times [0,2\pi]$$

$$\frac{\partial \vec{r}}{\partial u} = -r\sin u\cos v\,\vec{i} - r\sin u\sin v\,\vec{j} + r\cos u\,\vec{k}$$

$$\frac{\partial \vec{r}}{\partial v} = -(R + r\cos u)\sin v\,\vec{i} + (R + r\cos u)\cos v\,\vec{j}$$

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = |\cdots| = r(R + r\cos u)$$

$$\text{areal}(T) = \int_0^{2\pi}\int_0^{2\pi} r(R + r\cos u)\,du\,dv$$

$$= \int_0^{2\pi} \Big[rRu + r^2\sin u \Big]_0^{2\pi} dv$$

$$= \int_0^{2\pi} 2\pi rR\,dv = 4\pi^2 rR$$

$$= \underline{\underline{(2\pi r)(2\pi R)}}$$

Areal av grafer

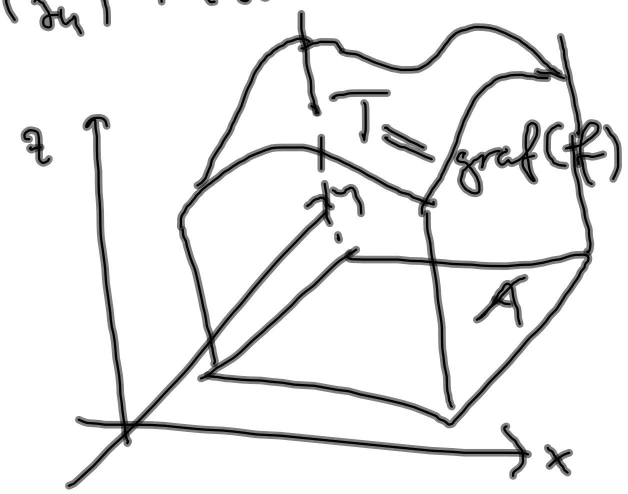
$$\vec{r}(u,v) = (u, v, f(u,v)) \in \mathbb{R}^3$$
$$= u\vec{i} + v\vec{j} + f(u,v)\vec{k}$$

$$f : A \longrightarrow \mathbb{R}$$

$$\frac{\partial \vec{r}}{\partial u}(u,v) = \left(1, 0, \frac{\partial f}{\partial u}(u,v)\right)$$
$$\frac{\partial \vec{r}}{\partial v}(u,v) = \left(0, 1, \frac{\partial f}{\partial v}(u,v)\right)$$
$$\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)(u,v) = \left(-\frac{\partial f}{\partial u}(u,v), -\frac{\partial f}{\partial v}(u,v), 1\right)$$
$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}$$

areal (grafen til f)

$$= \iint_A \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}\, du\, dv$$



Flateintegral av skalarfelt

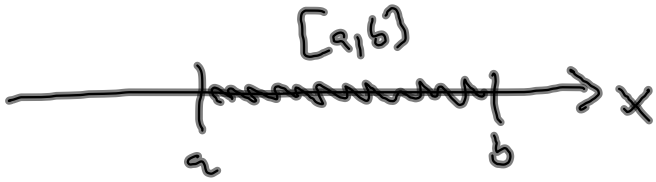
$$\begin{tikzcd} & A \arrow[r, "\vec{r}"] & T \subset \mathbb{R}^3 \arrow[dr, "f"] & \\ \mathbb{R}^2 \arrow[ur, phantom, "\supset" sloped] & & & \mathbb{R} \end{tikzcd}$$

$$\iint_T f \, dS = \iint_A f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|(u,v) \, du\,dv$$

LH 6.5 Greens teorem

Analysens fundamentalteorem

$$\Big[F(x)\Big]_{x=a}^{x=b} = \int_a^b \frac{dF}{dx}(x)\,dx$$



$\{a, b\}$

F

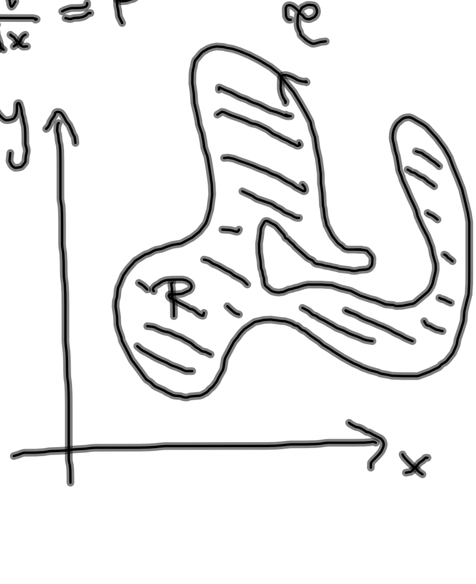
$[a, b]$

$\frac{dF}{dx} = F'$

$$\vec{F} = (P, Q)$$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

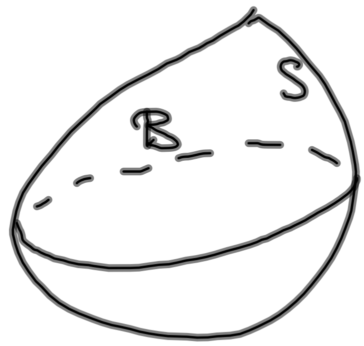


Stokes' teorem

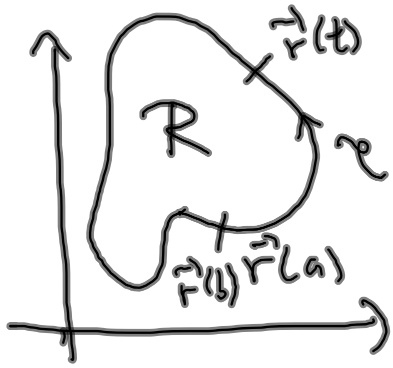
$$\iint_T \text{ vs } \iint_{\mathcal{C}}$$

Gauß' = Gauss' teorem

$$\iiint_B \quad \iint_S$$



Greens teorem



La $\mathcal{C}$ være en kurve i $\mathbb{R}^2$ parametrisert ved

$$\vec{r}(t) = (x(t), y(t))$$

$$t \in [a,b]$$

$\mathcal{C}$ er <u>lukket</u> hvis $\vec{r}(a) = \vec{r}(b)$

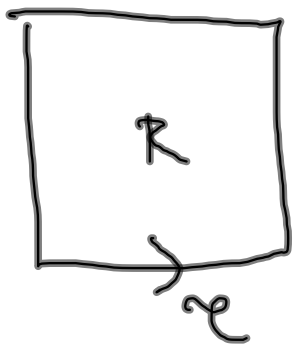
$\mathcal{C}$ en <u>enkel</u> hvis $\vec{r}(s) \neq \vec{r}(t)$ for $s \neq t$ i $[a,b)$.

<u>Jordans kurveteorem</u>: En enkel, lukket kurve $\mathcal{C}$ i $\mathbb{R}^2$ deler planet i to deler.

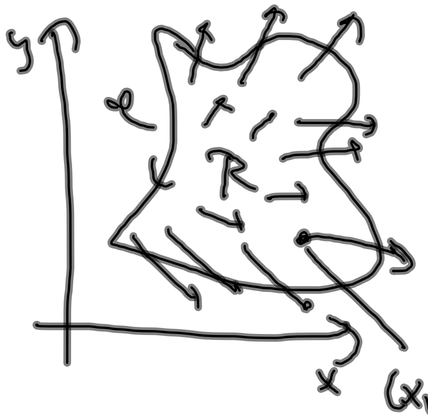
En begrenset del = innsiden R

En ubegrenset del = utsiden

Antar at $\vec{r}$ er <u>stykkevis glatt</u>.



Vektorfelt på $\mathcal{R}$



$$\mathcal{R} \xrightarrow{\vec{F}} \mathbb{R}^2$$

$$\vec{F}(x,y) = (P(x,y), Q(x,y))$$

$$\vec{F} = (P, Q)$$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b (P(x(t),y(t)), Q(x(t),y(t))) \cdot (x'(t), y'(t))\, dt$$

$$= \int_a^b \left(P(x(t),y(t))\, x'(t) + Q(x(t),y(t))\, y'(t)\right) dt$$

$$= \underbrace{\int_a^b}_{\mathcal{C}} \underbrace{P(x(t),y(t))}_{P}\, \underbrace{x'(t)\,dt}_{dx} + \underbrace{Q(x(t),y(t))}_{Q}\, \underbrace{y'(t)\,dt}_{dy}$$

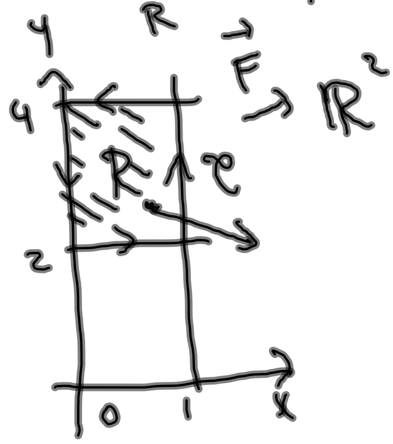
$$= \int_{\mathcal{C}} P\, dx + Q\, dy$$

Teorem La $\mathcal{C}$ være en stykkevis glatt, enkel, lukket kurve i $\mathbb{R}^2$; la R være området avgrenset av $\mathcal{C}$; la $P, Q : R \to \mathbb{R}$ ha kontinuerlige partielle deriverte.

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

der $\mathcal{C}$ er orientert mot klokken

<u>Eks</u> 1 Bestemmer $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ vha $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$



$$\vec{F}(x,y) = \left(x + \frac{y^3}{3},\ 2x + y^2\right)$$

$$P = P(x,y) = x + y^3/3$$

$$Q = Q(x,y) = 2x + y^2$$

$$\frac{\partial Q}{\partial x} = 2 \qquad \frac{\partial P}{\partial y} = y^2$$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} P\,dx + Q\,dy$$

$$\overset{\text{Green!}}{=} \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy = \int_0^1 \int_2^4 (2 - y^2)\, dy\, dx$$

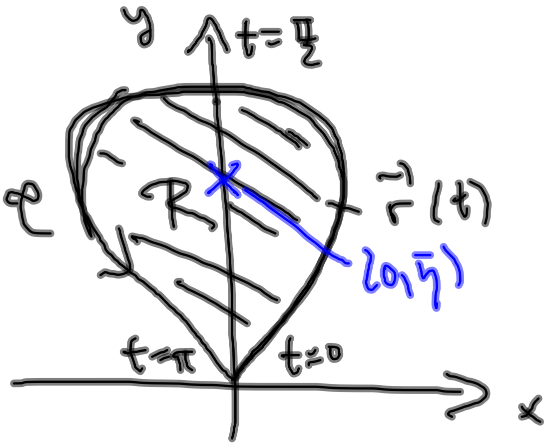
$$= \underline{\underline{-44/3}}$$

<u>Eks 2</u> $\iint_R f(x,y)\,dxdy$ ved å skrive

$$f = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\mathcal{C}$ og beregne $\int_{\mathcal{C}} P\,dx + Q\,dy$

$$\vec{r}(t) = (\sin t \cos t,\ \sin t) \qquad t \in [0, \pi]$$



areal(R)

$$\overset{\shortparallel}{|R|} = \iint_R 1\,dxdy$$

$$|R|\bar{y} = \iint_R y\,dxdy$$