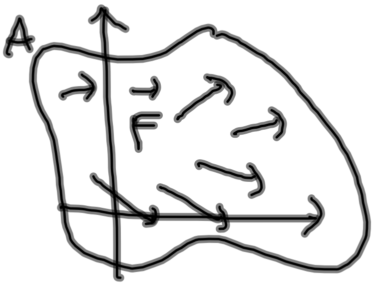
Litt mer om gradientfelt.



$A \subset \mathbb{R}^n$ område

$\phi : A \longrightarrow \mathbb{R}$ skalarfelt

$\boxed{\vec{F} = \nabla \phi}$ vektorfelt

$\vec{F} : A \longrightarrow \mathbb{R}^n$ $\quad \vec{F}(\vec{x}) = \left(\frac{\partial \phi}{\partial x_1}(\vec{x}), \ldots, \frac{\partial \phi}{\partial x_n}(\vec{x})\right)$

$\vec{F}$ er et gradientfelt/ er konservativt

$\mathcal{C}$ kurve i A $\quad \vec{r} : [a, b] \longrightarrow A \subset \mathbb{R}^n$

parametrisering

$$\boxed{\begin{aligned} \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} \nabla \phi \cdot d\vec{r} &= \phi(\vec{r}(b)) - \phi(\vec{r}(a)) \\ &= \phi(\vec{b}) - \phi(\vec{a}) \end{aligned}}$$

$\vec{a} = \vec{r}(a), \quad \vec{b} = \phi(b)$

Hvordan gjenkjenne gradientfelt?

Vektorfelt $\vec{F}: A \to \mathbb{R}^n$ der $A \underset{\text{åpent}}{\subseteq} \mathbb{R}^n$

med kontinuerlige $\frac{\partial F_i}{\partial x_j}$ for alle i, j.

To egenskaper

(1) $\vec{F}$ er <u>konservativt</u> $(\exists \phi : \vec{F} = \nabla \phi)$

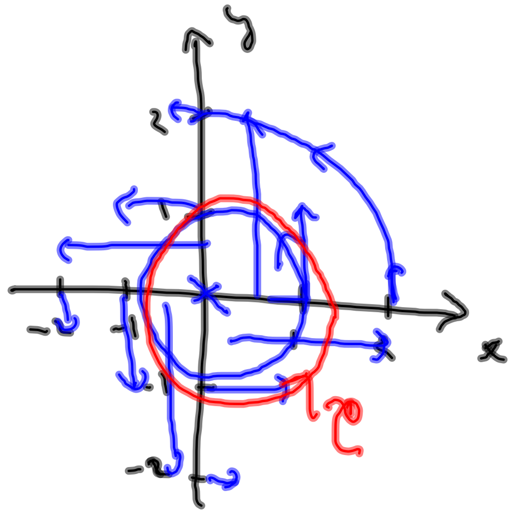
(2) $\vec{F}$ er <u>lukket</u>

$$\left(\frac{\partial F_i}{\partial x_j}(\vec{x}) = \frac{\partial F_j}{\partial x_i}(\vec{x}) \text{ for alle } \vec{x} \in A \right)$$

<u>SETN. 3.5.3</u> konservativt $\Longrightarrow$ lukket

Det omvendte gjelder ikke generelt.

$A = \mathbb{R}^2 \setminus \{(0,0)\}$



Eks.

$$\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

$$F_1(x,y) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial F_1}{\partial y}(x,y) = \frac{(-1)(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2} \stackrel{!}{=} \frac{\partial F_2}{\partial x}(x,y)$$

$\therefore$ $\vec{F}$ er lukket: $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ i hele A.

Men $\vec{F}$ er ikke konservativt. $(\vec{F} \neq \nabla \phi)$

La $\mathcal{C}$ være enhetssirkelen parametrisert

ved $\vec{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi)$

med $\vec{v}(t) = \vec{r}\,'(t) = (-\sin t, \cos t)$

og

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{\vec{F}(\cos t, \sin t)}_{= \left(\frac{-\sin t}{1}, \frac{\cos t}{1} \right)} \cdot (-\sin t, \cos t)\, dt$$

$$= \int^{2\pi} \sin(t)^2 + \cos(t)^2 \, dt = \int_0^{2\pi} 1 \, dt = 2\pi. \neq 0$$

TEOREM 3.5.4

La $A \subseteq \mathbb{R}^n$ være et åpen, enkeltsammenhengende område. La $\vec{F} : A \to \mathbb{R}^n$ ha kontinuerlige partielle deriverte $(\partial F_i / \partial x_j)$.

Da er $\vec{F}$ konservativt $(\vec{F} = \nabla \phi)$ hvis og bare hvis $\vec{F}$ er lukket $(\partial F_i / \partial x_j = \partial F_j / \partial x_i$ i hele A for alle $1 \le i,j \le n.)$

LH 3.6 : Kjeglesnitt

= ellipser, parabler og hyperbler

Kjegle i rommet : $z^2 = x^2 + y^2$

Plan i rommet : $z = ax + c$

Skjærer hverandre der

$$(ax+c)^2 = x^2 + y^2$$

Får

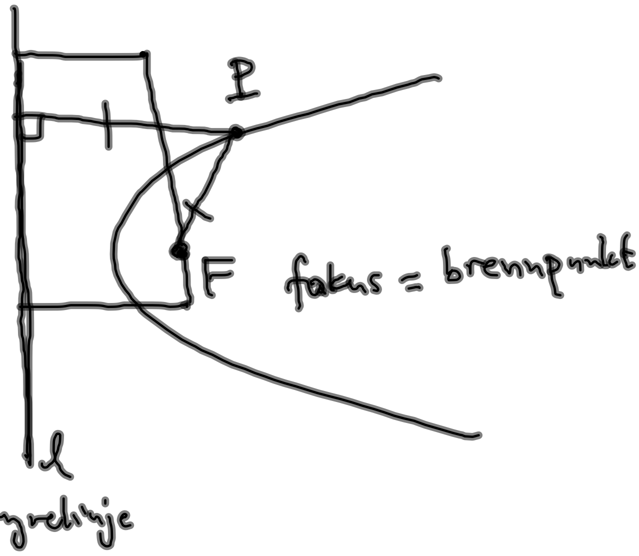
$|a| < 1$: ellipse

$|a| = 1$: parabel

$|a| > 1$: hyperbel

Geometriske definisjoner av kjeglesnitt som figurer i et plan:

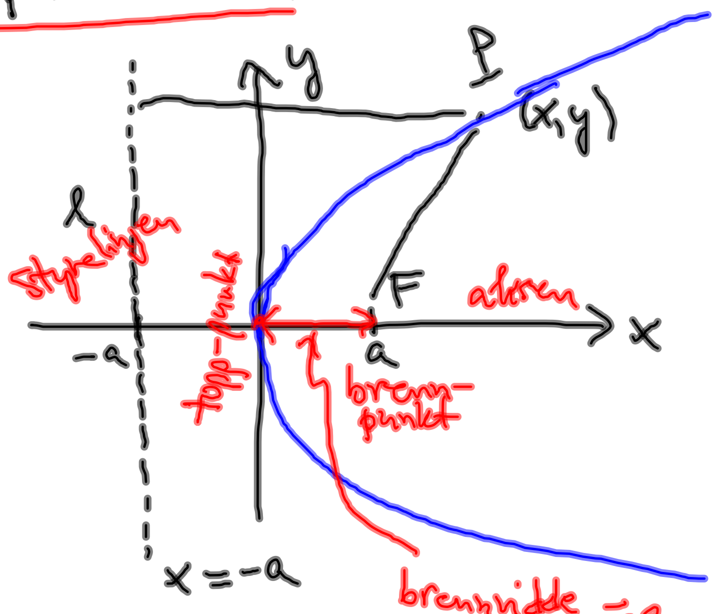
Parabel



Gitt en linje ℓ og et punkt F ser vi på punktene P der avstanden fra P til ℓ er lik avstanden fra P til F.

Kartesisk likning:

$a > 0$

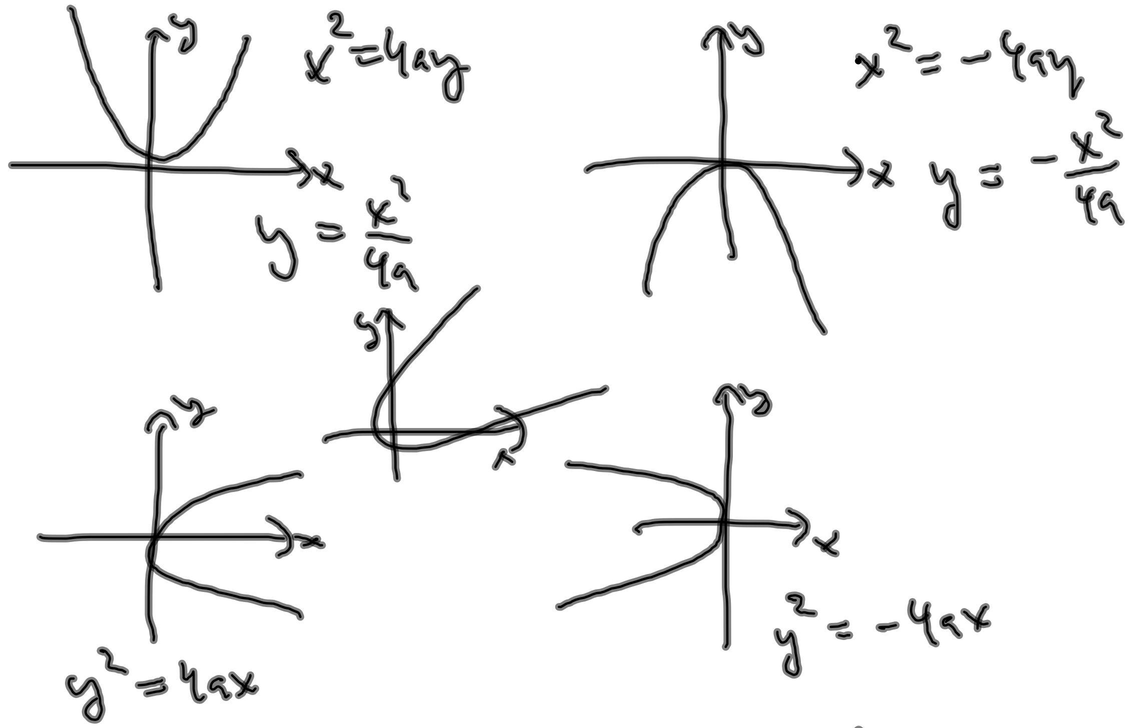


$$\boxed{x+a = \sqrt{(x-a)^2+y^2}}$$

(regne litt)

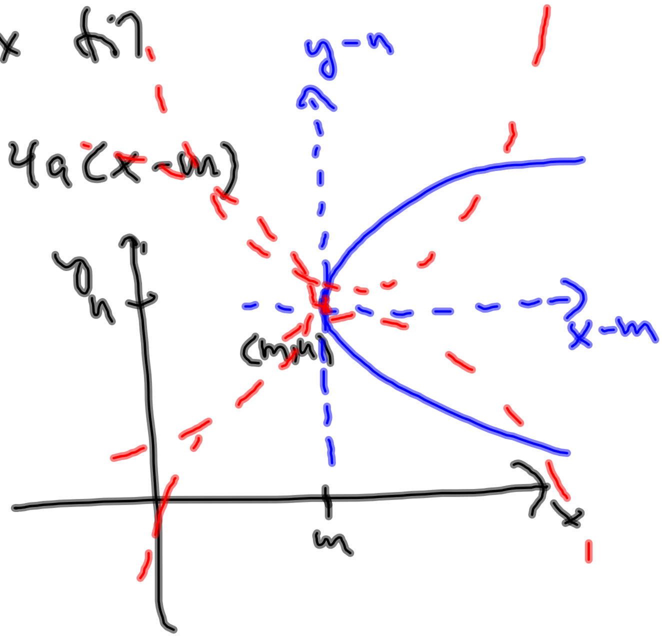
$$\boxed{y^2 = 4ax}$$

$$y = \pm\sqrt{4ax}$$



Hvis topp-punktet er i (m,n) endres likningen $y^2 = 4ax$ til

$$(y-n)^2 = 4a(x-m)$$



Eks. på opprydding

$$\boxed{x^2+2x} + y = 0$$

$$\Downarrow$$

$$(x+1)^2 - 1 + y = 0$$

$$(x+1)^2 = -(y-1)$$

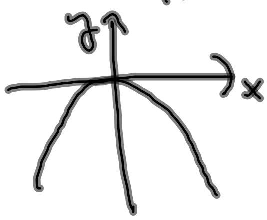
toppunkt i $(m,n) = (-1,1)$

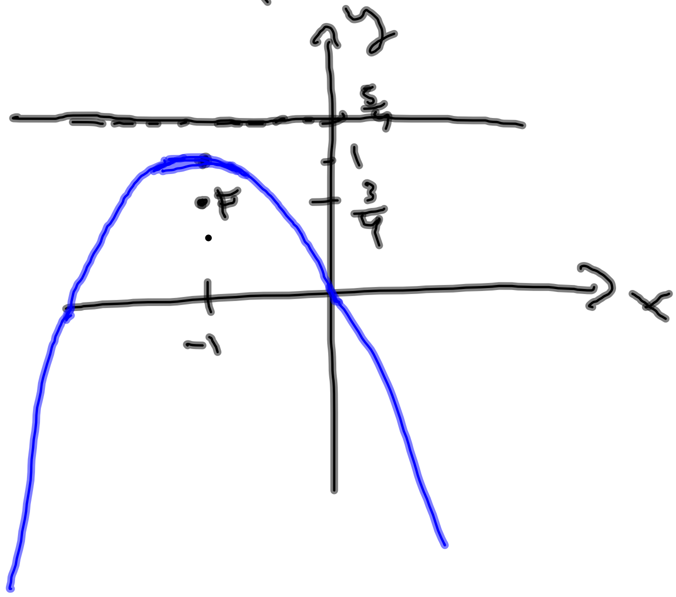
brennvidde $= \frac{1}{4}$

$$x^2 = 4ay$$

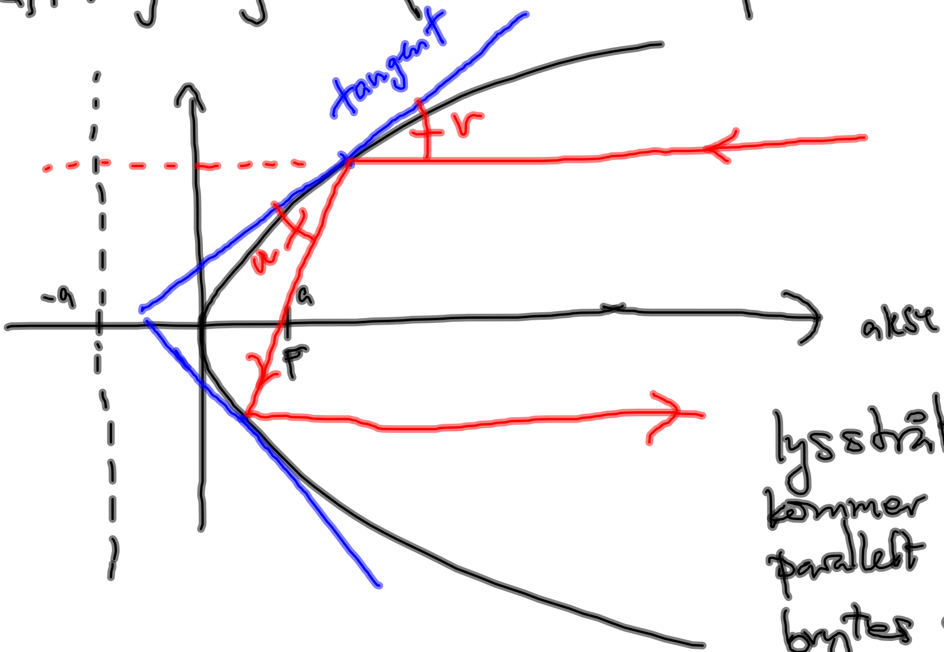
$$x^2 = -4ay$$

$$a = \frac{1}{4}.$$





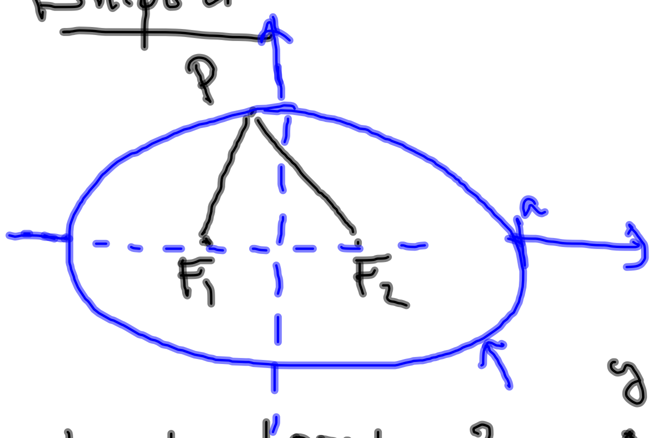
Refleksjonsegenskapen til en parabel:



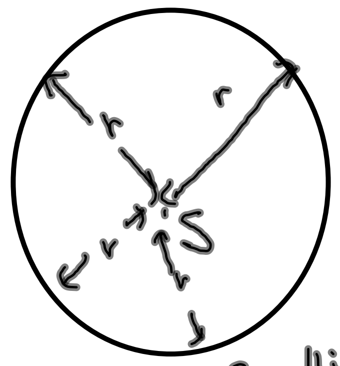
v = w

lysstråler som kommer inn parallelt med aksen brytes gjennom fokus = brennpunktet

Ellipser

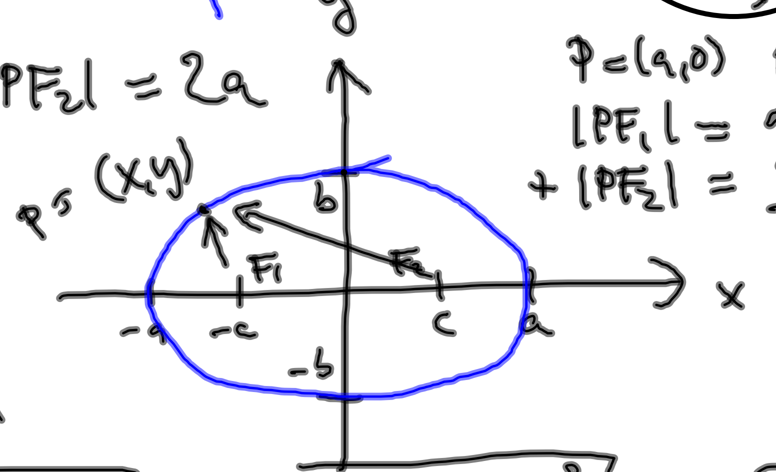


Sirkler



$$|PF_1| + |PF_2| = 2a$$

$F_1 = (-c, 0)$
$F_2 = (c, 0)$



$P = (a,0)$ på ellipsen

$$|PF_1| = a + c$$
$$+\ |PF_2| = a - c$$
$$= 2a$$

$|PF_1| + |PF_2|$

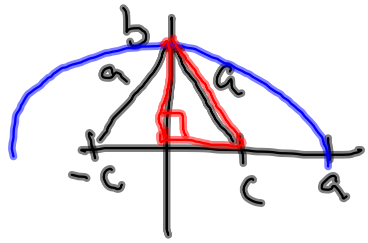
$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

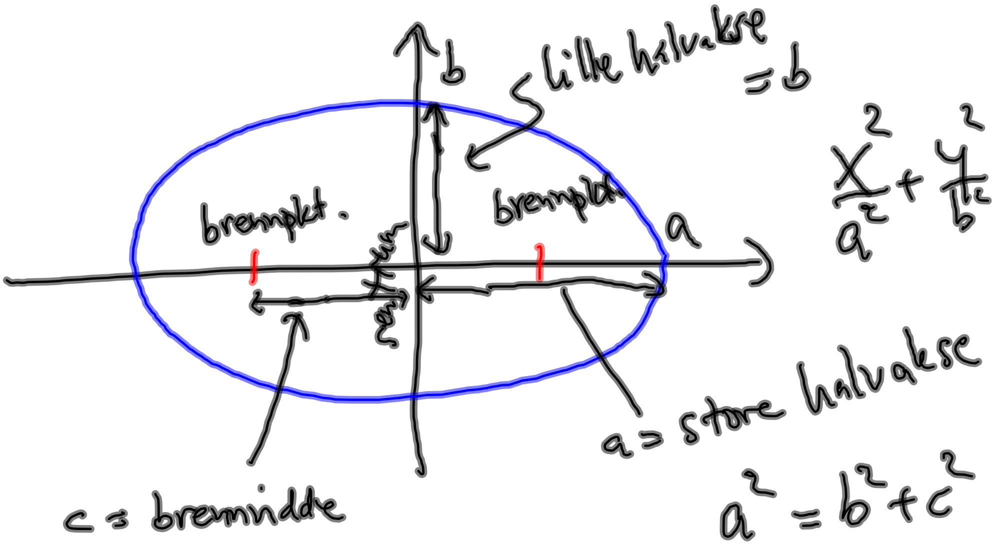
[regne litt]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

der

$$b^2 + c^2 = a^2$$
$$b = \sqrt{a^2 - c^2}.$$

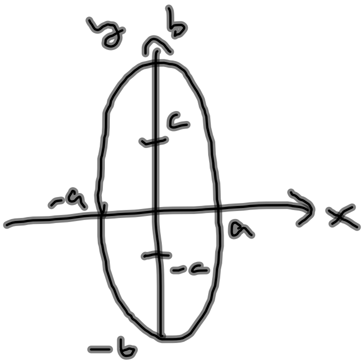




$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$a^2=b^2+c^2$$

$$a \geq b > 0$$



$$b \geq a > 0$$

$$b^2=a^2+c^2$$

Mer generelt:

$$\frac{(x-m)^2}{a^2}+\frac{(y-n)^2}{b^2}=1$$

er en ellipse med sentrum i (m,n), halvakser a og b, brennvidde

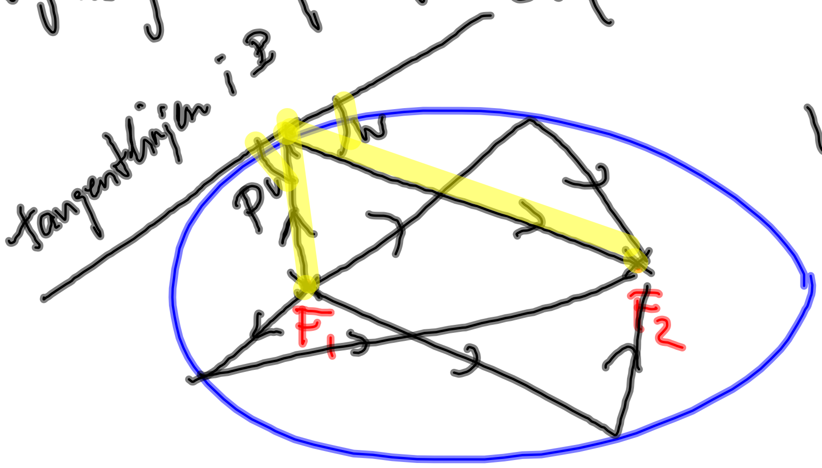
$$c^2=|a^2-b^2|$$

Parametrisering:

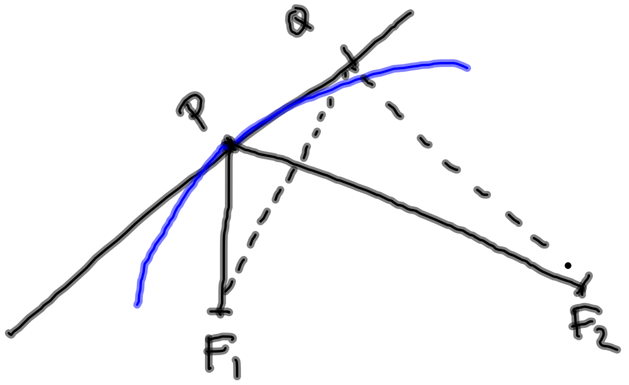
$$\vec{r}(t)=(m+a\cos t,\ n+b\sin t)$$

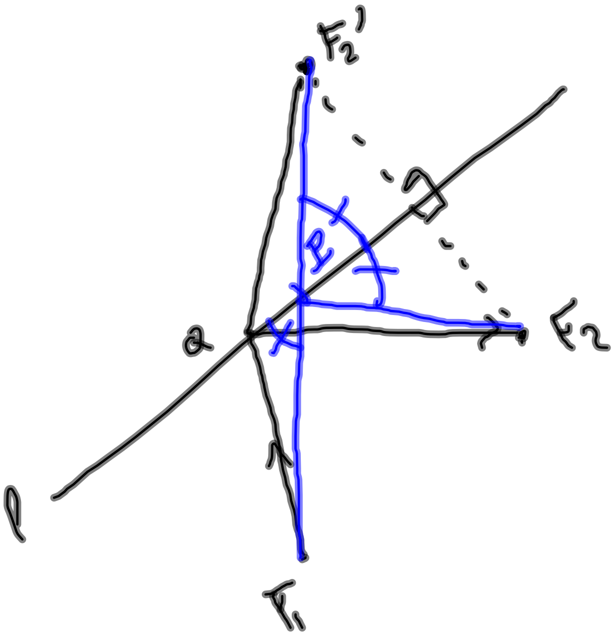
for $0 \leq t \leq 2\pi$ $\quad t\in[0,2\pi)$

Refleksjonsegenskapen for ellipsen



$v = w$





$|F_1Q| + |QF_2|$
minst
mulig