- Egenvektorer og -verdier
- Affine avbildninger
- Determinanter som skaleringsfaktor

Egenvektorer

$$\vec{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$n \times n$ matrise A $\quad \vec{T}(\vec{x}) = A\vec{x}$

Ser etter vektoren $\vec{v}$ slik at

$$T(\underline{\vec{v}}) = \underline{\lambda}\,\underline{\vec{v}} = A\underline{\vec{v}}$$

Definisjon La A være en $n\times n$-matrise. En vektor $\vec{v} \in \mathbb{R}^n$, med $\vec{v} \neq \vec{0}$, er en egenvektor for A hvis

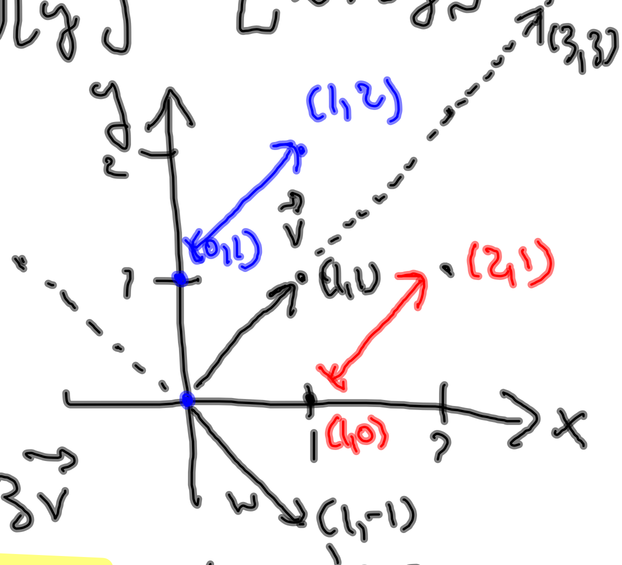
$$A\vec{v} = \lambda \vec{v}$$

for en skalar $\lambda \in \mathbb{R}$. Vi kaller λ egenverdien til $\vec{v}$.

Eksempel: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$\vec{T}(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ x+2y \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$



$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{v}$$

$\vec{v}$ er en egenvektor for A, med egenverdi $\lambda = 3$.

$$A\vec{w} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{w}$$

$\vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ er også en egenvektor for A, med $\lambda = 1$.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "T"] \arrow[rr, bend right, "T^2"'] \arrow[rrr, bend right=50, "T^3"'] & \mathbb{R}^n \arrow[r, "T"] & \mathbb{R}^n \arrow[r, "T"] & \mathbb{R}^n \; \cdots
\end{tikzcd}$$

$$\vec{x} \longmapsto A\vec{x} \longmapsto AA\vec{x} \longmapsto A^3\vec{x} \; \cdots$$
$$= A^2\vec{x}$$

l gjentagelser

$$\vec{x} = A^l \vec{x}$$

Hvis $A\vec{v} = \lambda \vec{v}$

$$\vec{v} \longmapsto \lambda\vec{v} \longmapsto \lambda^2\vec{v} \longmapsto \lambda^3\vec{v} \; \cdots$$

-er $A^l \vec{v} = \lambda^l \vec{v}$

La $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ være egenvektorer for A, med egenverdier $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Da er $A^l \vec{v}_i = \lambda_i^l \vec{v}_i$

Hvis $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_k \vec{v}_k$

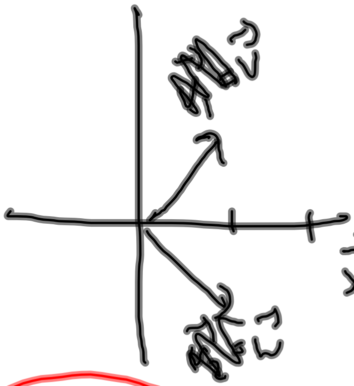
er
$$A^l \vec{x} = A^l (c_1 \vec{v}_1 + \ldots + c_k \vec{v}_k)$$
$$= c_1 A^l \vec{v}_1 + \ldots + c_k A^l \vec{v}_k$$

$$\boxed{A^l \vec{x} = c_1 \lambda_1^l \vec{v}_1 + \ldots + c_k \lambda_k^l \vec{v}_k}$$

$$\vec{x} = \vec{v} + \vec{w}$$

$$A\vec{v} = 3\vec{v}$$

$$A\vec{w} = \vec{w}$$



$$\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

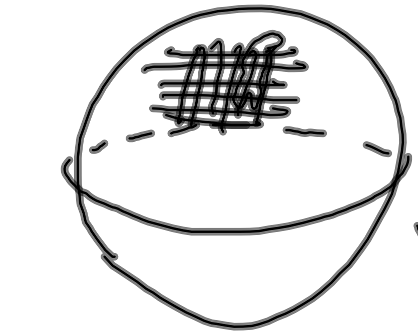
$$A^{\ell}\vec{x} = 3^{\ell}\vec{v} + 1^{\ell}\vec{w}$$

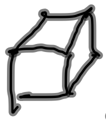
$$= 3^{\ell}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3^{\ell}+1 \\ 3^{\ell}-1 \end{bmatrix}.$$

$$\vec{x} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad A\vec{x} = \begin{bmatrix} 3+1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^2 x = \begin{bmatrix} 9+1 \\ 9-1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

Affinavbildninger





$1\,\text{km}^3$

$n = 6 \times 10^9$

$\vec{v} \in \mathbb{R}^{6000000000} = \mathbb{R}^n$

$\vec{F}$

$\vec{v} \longmapsto \vec{F}(\vec{v})$

$\vec{a}$

$$\vec{F}(\vec{a}+\vec{r}) = \boxed{\underbrace{\vec{F}(\vec{a})}_{\text{konstant}} + (\text{noe lineært i } \vec{r})} + \dots$$

Affinavbildninger

En funksjon $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$ kalles en **affin avbildning** hvis det finnes en $m \times n$-matrise A og en vektor $\vec{c} \in \mathbb{R}^m$ slik at

$$\boxed{\vec{F}(\vec{x}) = A\vec{x} + \vec{c}}$$

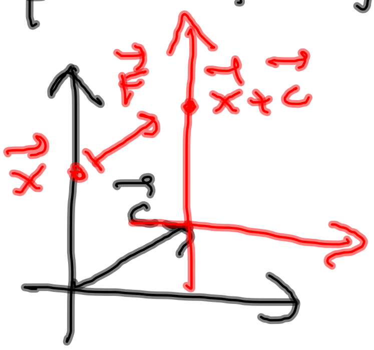
Eks $\vec{c} = \vec{0}$: $\vec{x} \mapsto A\vec{x}$ lineær

Eks $A = I_n$ $(m = n)$ $\quad I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$$\vec{x} \longmapsto I_n \vec{x} + \vec{c} = \vec{x} + \vec{c}$$

parallellforskyvning

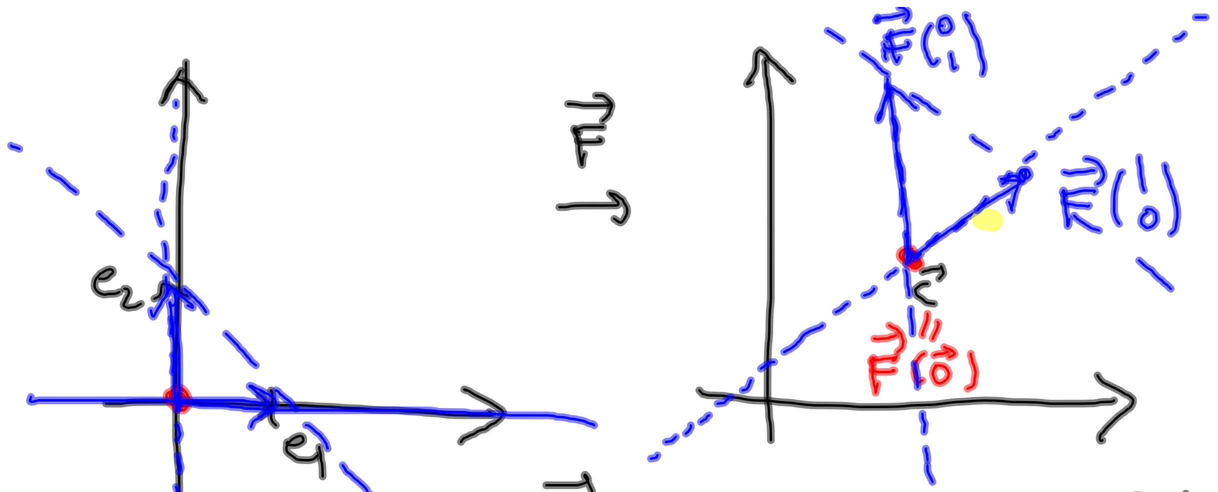
$$\begin{tikzcd} \vec{x} \arrow[r, mapsto, "A"] \arrow[dr, mapsto] & A\vec{x} \arrow[d, mapsto, "+\vec{c}"] \\ & A\vec{x} + \vec{c} \end{tikzcd}$$



$f : \mathbb{R} \to \mathbb{R}$

$f(x) = ax$ lineær

$f(x) = ax + c$ affine



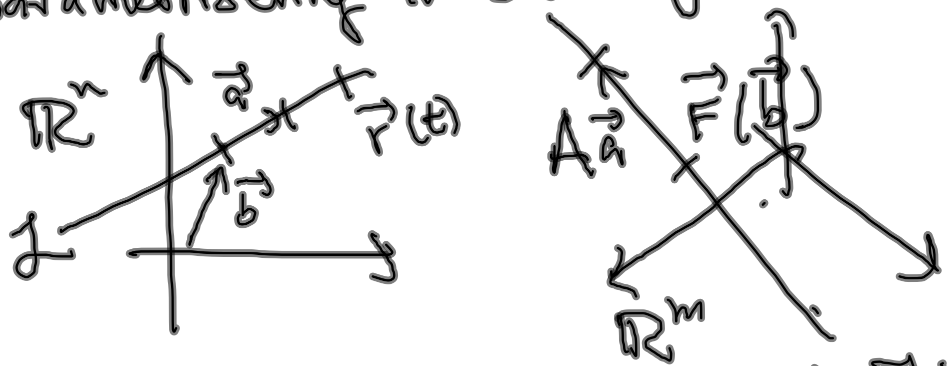
$$\vec{x} \overset{\vec{F}}{\longmapsto} A\vec{x} + \vec{c} \qquad \vec{F}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\a_{12}\end{pmatrix} + \vec{c}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \vec{F}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a_{12}\\a_{22}\end{pmatrix} + \vec{c}$$

Setning 1.10.2 $\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ affin

$$\vec{F}(\vec{x}) = A\vec{x} + \vec{c}$$

La $\vec{r}(t) = t\vec{a} + \vec{b}$ være en parametrisering av en linje L i $\mathbb{R}^n$



Da er $\vec{F}(\vec{r}(t)) = A(t\vec{a} + \vec{b}) + \vec{c}$

$$= tA\vec{a} + \underline{A\vec{b} + \vec{c}}$$

$$= t A\vec{a} + \vec{F}(b)$$

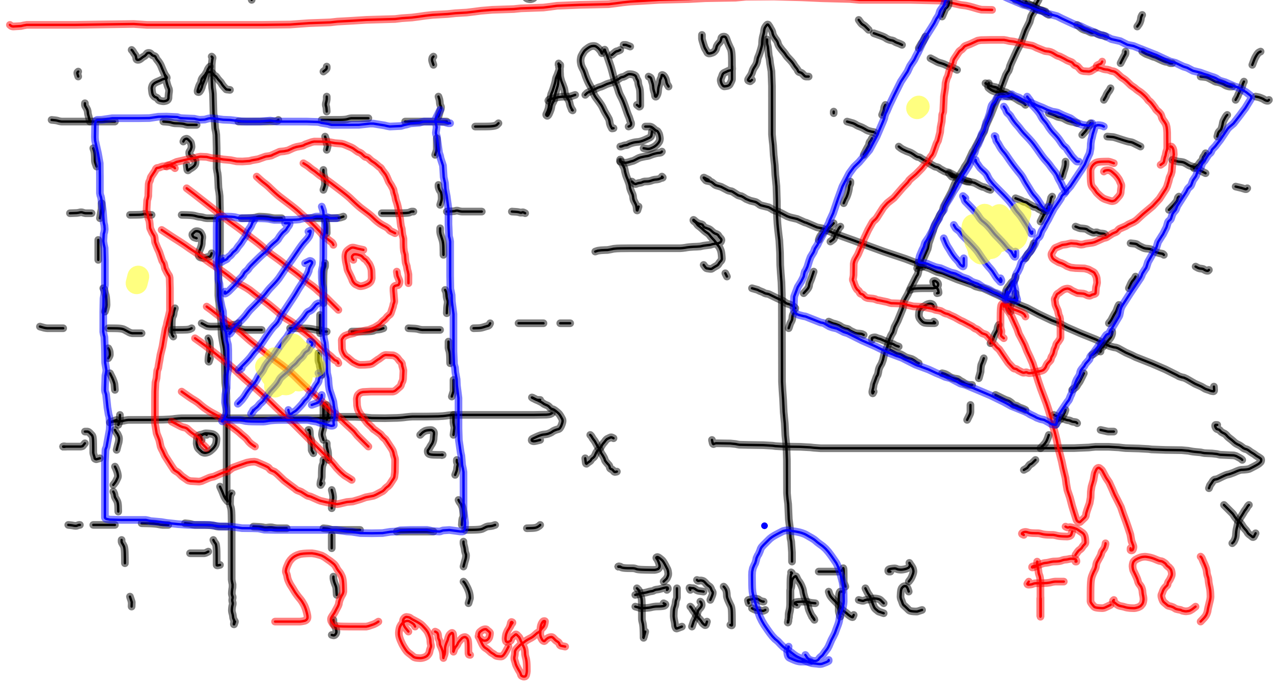
så $\vec{F}(L)$ er linjen i $\mathbb{R}^m$

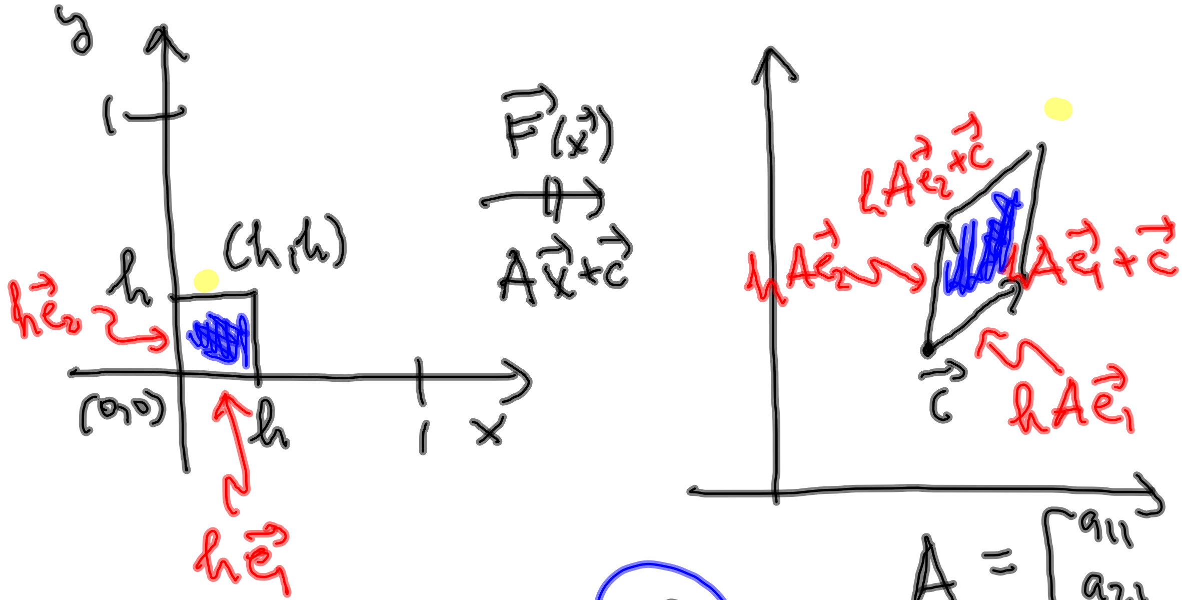
gjennom $\vec{F}(\vec{b}) = A\vec{b} + \vec{c}$

som er parallell med $A\vec{a}$.

NB: Gjelder bare hvis $A\vec{a} \neq \vec{0}$.

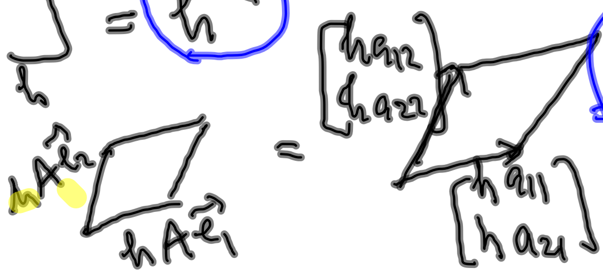
Areal, volum og determinant.





$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Arealet til $h\square_h = h^2$

Arealet til 

$= h^2|\det A|$

$$= \left|\begin{vmatrix} ha_{11} & ha_{12} \\ ha_{21} & ha_{22} \end{vmatrix}\right| = |ha_{11}\cdot ha_{22} - ha_{12}\cdot ha_{21}|$$

$$= h^2|\underbrace{a_{11}a_{22}-a_{12}a_{21}}_{\substack{\det(A)\\ =|A|}}| = h^2|\det(A)|$$

<u>Setning 9.10.3</u>

La $\vec{F}(\vec{x}) = A\vec{x} + \vec{c}$ være en affinavbildning $\mathbb{R}^2 \to \mathbb{R}^2$.

La $\Omega \subset \mathbb{R}^2$ være et „pent" område.

Da er

$$\boxed{\text{Areal}(\vec{F}(\Omega)) = |\det(A)| \cdot \text{Areal}(\Omega)}$$

skaleringsfaktor for areal

Hvis $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ og $\Omega \subset \mathbb{R}^3$

er

$$\text{Volum}(\vec{F}(\Omega)) = |\det(A)| \cdot \text{Volum}(\Omega)$$
