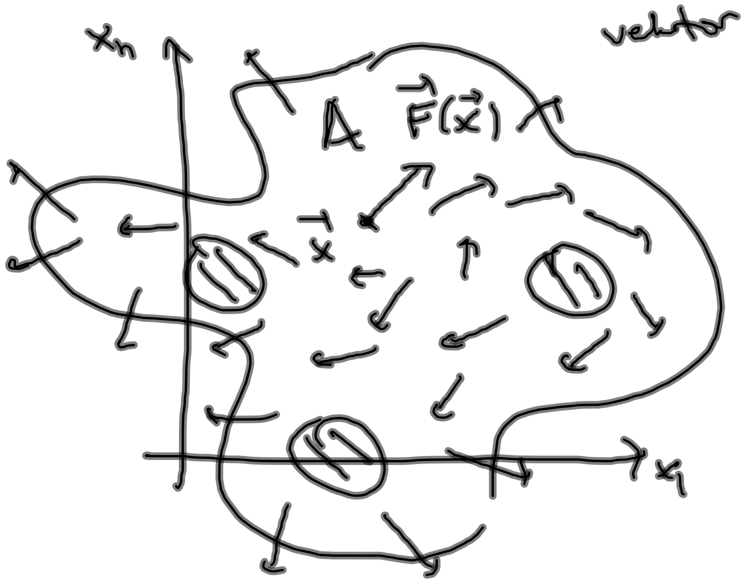
LH 3.8: Grafisk fremstilling av vektorfelt

$A \subset \mathbb{R}^n$ $\quad \vec{F} : A \longrightarrow \mathbb{R}^n$

tenker på $\vec{F}(\vec{x})$ som en vektor i $\vec{x}$



$\vec{F}$

$\longrightarrow \mathbb{R}^n$

Strømlinjer

$\vec{r} : [a,b] \to A$

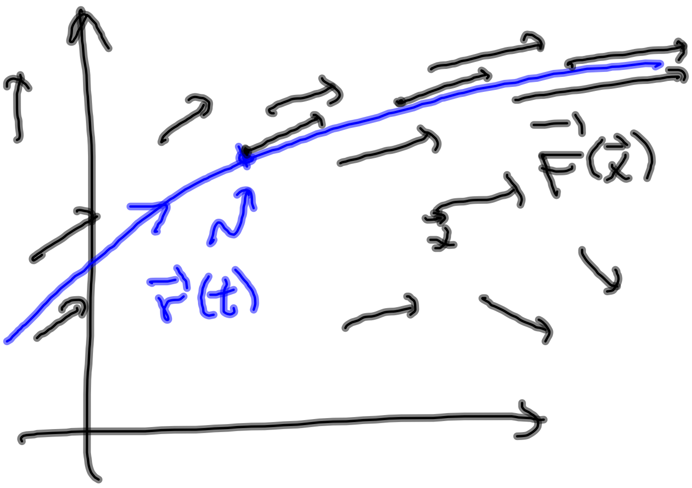
er en

strømlinje

for

$\vec{F} : A \to \mathbb{R}^n$

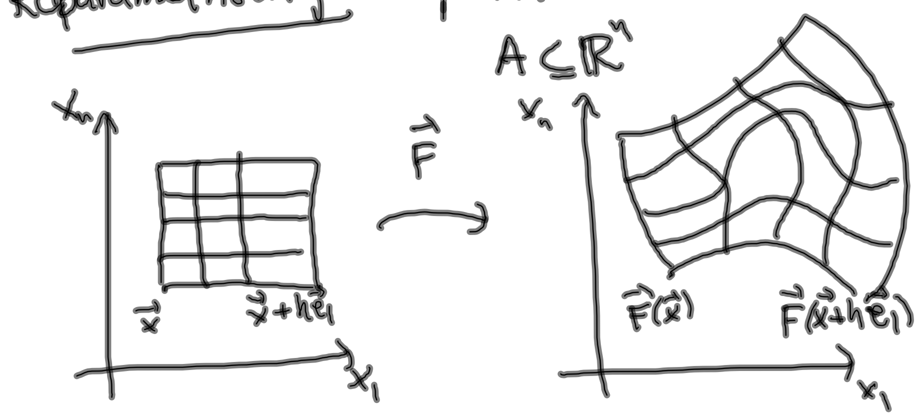
hvis $\vec{v}(t) = \boxed{\vec{r}\,'(t) = \vec{F}(\vec{r}(t))}$



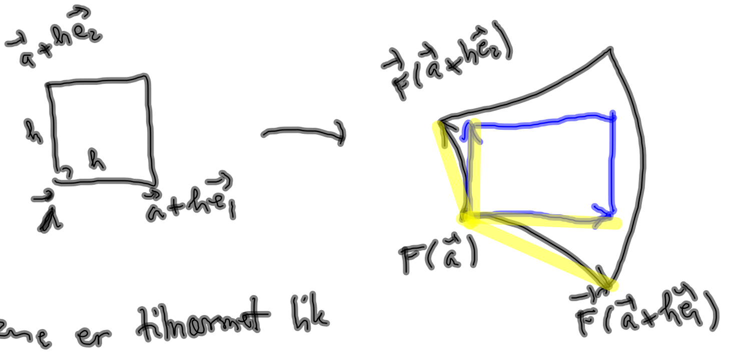
Reparametriseringer

$\vec{F}: A \to \mathbb{R}^n$

$A \subseteq \mathbb{R}^n$



$n=2$
areal



Endringene er tilnærmet lik

$$\vec{F}(\vec{a}+h\vec{e}_1) - \vec{F}(\vec{a}) \approx \vec{F}'(\vec{a})\, h\vec{e}_1 = h \begin{bmatrix} \partial F_1/\partial x_1 (\vec{a}) \\ \partial F_2/\partial x_1 (\vec{a}) \end{bmatrix}$$

$$F(\vec{a}+h\vec{e}_2) - \vec{F}(\vec{a}) \approx \vec{F}'(\vec{a})\, h\vec{e}_2 = h \begin{bmatrix} \partial F_1/\partial x_2 (\vec{a}) \\ \partial F_2/\partial x_2 (\vec{a}) \end{bmatrix}$$

Arealet til parallellogrammet utspent av

$\vec{F}'(\vec{a})\, h\vec{e}_1$ og $\vec{F}'(\vec{a})\, h\vec{e}_2$

er

$$|\det(\vec{F}'(\vec{a}))| \cdot h^2$$

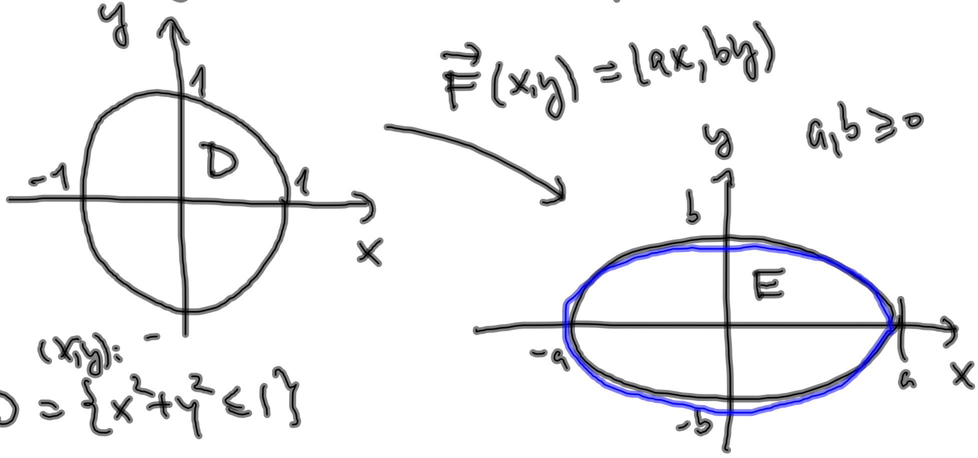
som er $\boxed{|\det \vec{F}'(\vec{a})|}$ ganger

arealet til kvadratet utspent av

$h\vec{e}_1$ og $h\vec{e}_2$.

Areal og omkrets av ellipser

$\vec{F}(x,y) = (ax, by)$ $\quad a, b \geq 0$



$D = \{(x,y) : x^2 + y^2 \leq 1\}$

$$E = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$$

$$\vec{F}'(\vec{a}) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad \det \vec{F}'(\vec{a}) = ab \geq 0$$

$$\text{areal}(E) = ab \ \text{areal}(D) = \pi a b$$

(Når $a = b = r$ er dette πr^2)

Hva er omkretsen til E?

$\vec{r}(t) = (a\cos t, b\sin t) \quad 0 \leq t \leq 2\pi$

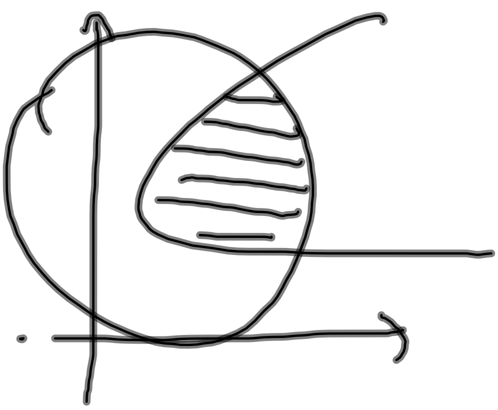
$v(t) = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$

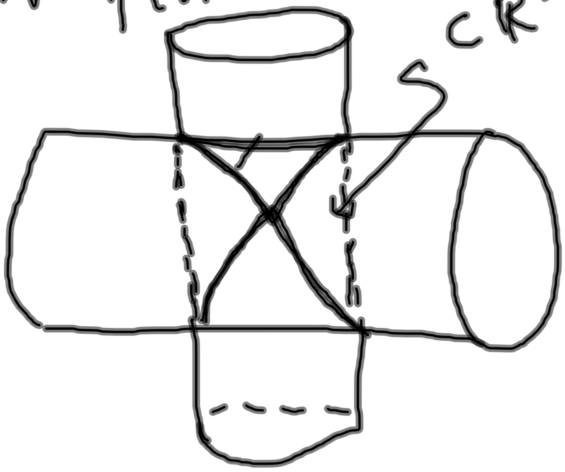
$$\text{omkrets} = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, dt = ?$$

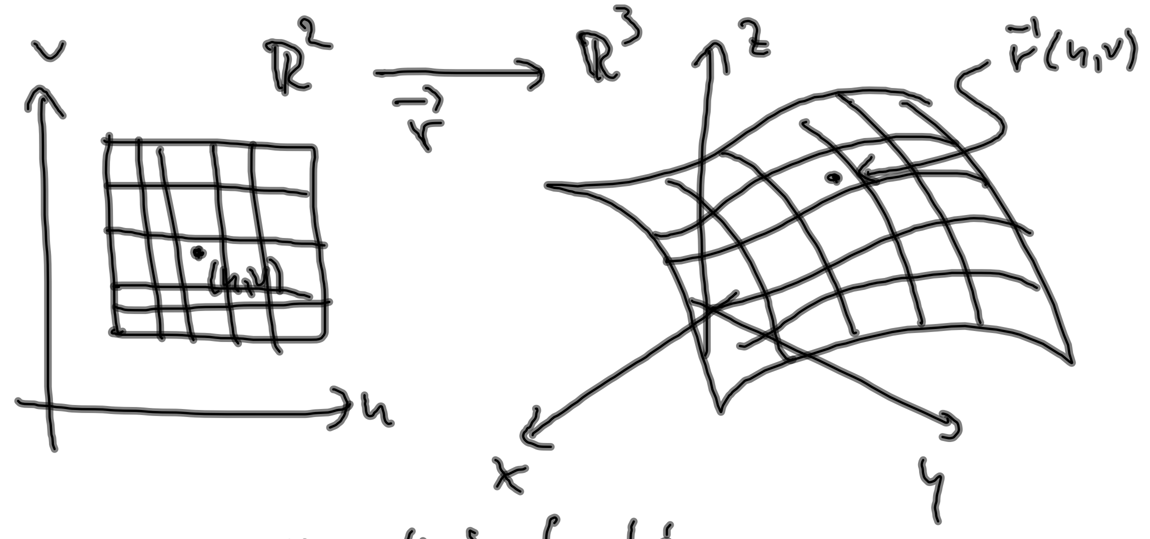
elliptisk integral
omvendt funksjon:
elliptisk funksjon
N. H. Abel

$(a = b = r : 2\pi r)$

LH 3.9 Parametriserte flater







En (kontinuerlig) funksjon

$$\vec{r} : A \longrightarrow \mathbb{R}^3$$

der $A \subseteq \mathbb{R}^2$ kalles en <u>parametrisering</u>

av flaten

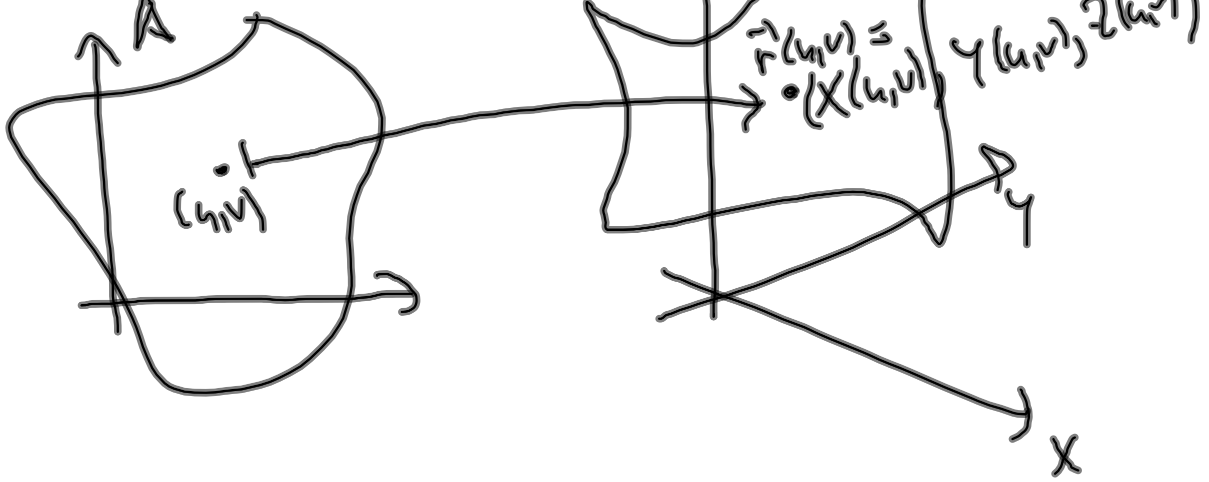
$$\{ \vec{r}(\vec{a}) \mid \vec{a} \in A \} = \vec{r}(A) \subset \mathbb{R}^3$$

$$\parallel$$

$$\{ \vec{r}(u,v) \mid (u,v) \in A \}$$

Skiver

$$\vec{r}(u,v) = X(u,v)\vec{i} + Y(u,v)\vec{j} + Z(u,v)\vec{k}$$



<u>Eks</u> Kuleflaten med radius R om $(0,0,0)$ kan parametriseres som to grafer:

$$A = \{(x,y) \mid x^2+y^2 \le R^2\}$$

$$f(x,y) = \sqrt{R^2-x^2-y^2}$$

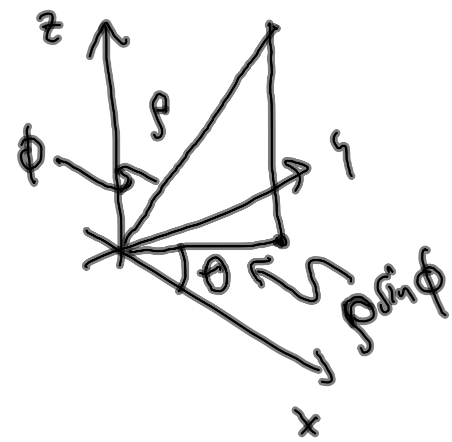
$$\vec{r} : A \to \mathbb{R}^3$$

$$\vec{r}(x,y) = (x, y, f(x,y)) = (x, y, \sqrt{R^2-x^2-y^2})$$

pluss

$$\vec{s}(x,y) = (x, y, -f(x,y))$$

ved kulekoordinater



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

kulen med radius R:

$\rho = R$

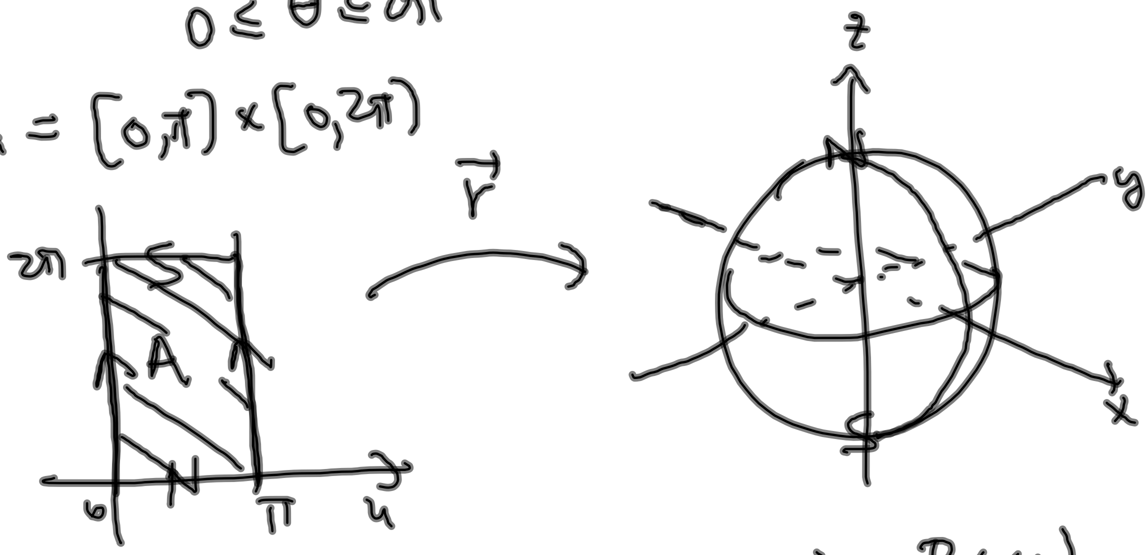
$0 \leq \phi \leq \pi$

$0 \leq \theta \leq 2\pi$

$u = \phi \in [0, \pi]$

$v = \theta \in [0, 2\pi]$

$A = [0, \pi] \times [0, 2\pi]$

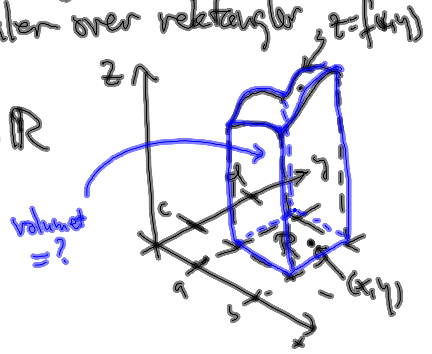


$$\vec{r}(u,v) = (\underbrace{R\sin u \cos v}_{x(u,v)}, \underbrace{R \sin u \sin v}_{y(u,v)}, \underbrace{R\cos u}_{z(u,v)})$$

LH 6: Integrasjon

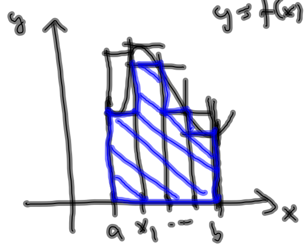
6.1 Dobbeltintegraler over rektangler

$f : \underbrace{[a,b]\times[c,d]}_{R} \to \mathbb{R}$



Rep.

$f : [a,b] \to \mathbb{R}$



partisjon P

$a = x_0 < x_1 < \dots < x_n = b$

$[x_{i-1}, x_i]$

lar $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$

og $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$

der $1 \le i \le n$. $\quad m_i \le M_i$

Nedre trappesum

$$N(P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$N(P) \le \varnothing(P)$

Øvre trappesum

$$\varnothing(P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

Nedreintegralet

$$\underline{\int_a^b} f(x)\,dx = \sup_{P} N(P)$$

$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$

Øvreintegralet

$$\overline{\int_a^b} f(x)\,dx = \inf_{P} \varnothing(P)$$

Hvis $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

sier vi at f er <u>integrerbar</u>

og lar $\int_a^b f(x)\,dx$ være den felles verdien.

Tilbake til $f : R \longrightarrow \mathbb{R}$ fra

$R = [a,b] \times [c,d]$

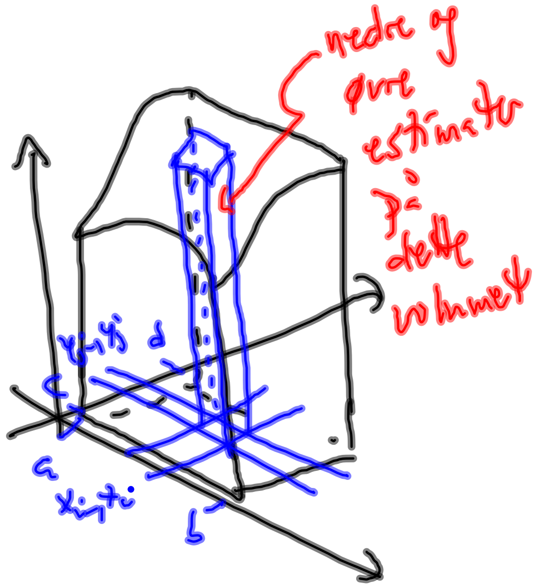
En <u>partisjon</u> Π av R

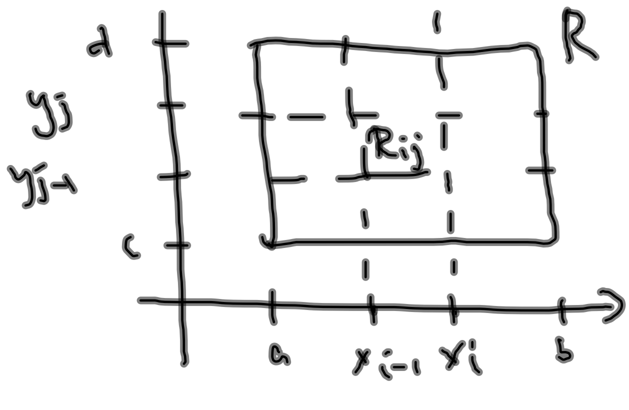
er en partisjon

$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$

og en partisjon

$c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d$





$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

$1 \leq i \leq n$

$1 \leq j \leq m$

Anta $f: R \to \mathbb{R}$ er begrenset.

La

$$m_{ij} = \inf\{ f(x,y) \mid (x,y) \in R_{ij} \}$$

$m_{ij} \le M_{ij}$

$$M_{ij} = \sup\{ f(x,y) \mid (x,y) \in R_{ij} \}$$

La nedre trappesum være

$$N(\Pi) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$

og la øvre trappesum være $N(\Pi) \le \text{Ø}(\Pi)$

$$\text{Ø}(\Pi) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$

La nedreintegralet være

$$\underline{\iint_R} f(x,y)\,dx\,dy = \sup_{\Pi} N(\Pi)$$

$\underline{\iint_R} f \le \overline{\iint_R} f$

og la øvreintegralet være

$$\overline{\iint_R} f(x,y)\,dx\,dy = \inf_{\Pi} \text{Ø}(\Pi)$$

Hvis $\underline{\iint_R} f = \overline{\iint_R} f$ sier vi at

f er integrerbar og lar

$$\iint_R f(x,y)\,dx\,dy$$

være den felles verdien.