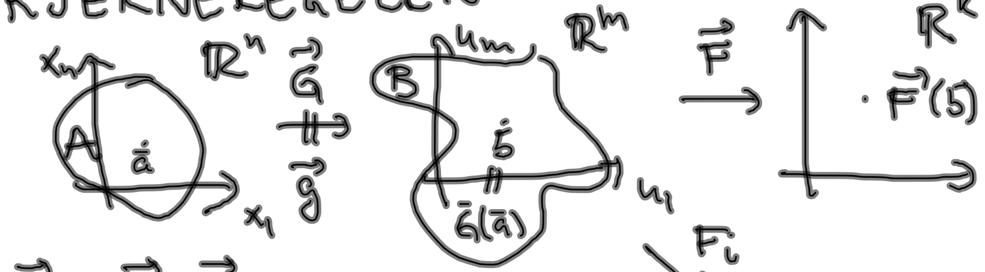
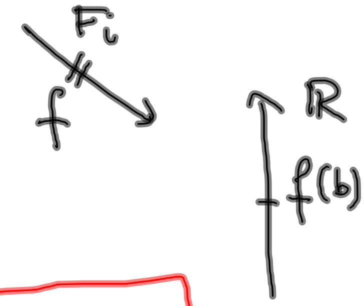
KJERNEREGELEN



$\vec{H} = \vec{F} \circ \vec{G}$



$$\vec{H}'(\vec{a}) = \vec{F}'(b) \cdot \vec{G}'(\vec{a})$$

$h = f \circ \vec{g}$

$$\nabla h(\vec{a}) = \nabla f(\vec{b}) \cdot \vec{g}'(\vec{a})$$

$$\nabla h(\vec{a}) = \left(\frac{\partial h}{\partial x_1}(\vec{a}), \dots, \frac{\partial h}{\partial x_n}(\vec{a})\right)$$

$$\nabla f(\vec{b}) = \left(\frac{\partial f}{\partial u_1}(\vec{b}), \dots, \frac{\partial f}{\partial u_m}(\vec{b})\right)$$

$$\vec{g}'(\vec{a}) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial g_1}{\partial x_n}(\vec{a}) \\ \vdots & & \\ \frac{\partial g_m}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial g_m}{\partial x_n}(\vec{a}) \end{bmatrix}$$

$$\nabla f(\vec{b})\vec{g}'(\vec{a}) = \left(\frac{\partial f}{\partial u_1}(\vec{b}) \quad \frac{\partial f}{\partial u_m}(\vec{b})\right)\left(\quad\right)$$

j'te søyle/komponent

$1 \leq j \leq n$

$$\boxed{\frac{\partial h}{\partial x_j}(\vec{a}) = \left(\frac{\partial f}{\partial u_1}(\vec{b}) \dots \frac{\partial f}{\partial u_m}(\vec{b})\right) \begin{pmatrix} \frac{\partial g_1}{\partial x_j}(\vec{a}) \\ \vdots \\ \frac{\partial g_m}{\partial x_j}(\vec{a}) \end{pmatrix} = \frac{\partial f}{\partial u_1}(\vec{b})\frac{\partial g_1}{\partial x_j}(\vec{a}) + \dots + \frac{\partial f}{\partial u_m}(\vec{b})\frac{\partial g_m}{\partial x_j}(\vec{a})}$$

Eks

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\vec{g}"] \arrow[rr, bend right, "h"] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$f(u_1, u_2) = u_1^2 + u_2^2$

$\vec{g} = (g_1, g_2)$

$g_1(x_1, x_2) = x_1^2 - x_2^2$

$g_2(x_1, x_2) = 2x_1x_2$

$h(x_1, x_2) = f(g_1(x_1, x_2), g_2(x_1, x_2))$

$h(\vec{x}) = f(\vec{g}(\vec{x}))$

$$\boxed{\frac{\partial h}{\partial x_1}(\vec{a})} = \frac{\partial f}{\partial u_1}(\vec{b}) \cdot \frac{\partial g_1}{\partial x_1}(\vec{a}) + \frac{\partial f}{\partial u_2}(\vec{b}) \cdot \frac{\partial g_2}{\partial x_1}(\vec{a})$$

$\vec{a} = (x_1, x_2)$

$\vec{b} = (u_1, u_2)$

$$= 2u_1 \cdot 2x_1 + 2u_2 \cdot 2x_2$$

$$= 2(x_1^2 - x_2^2)2x_1 + 2(2x_1x_2)2x_2$$

$$= \boxed{4x_1^3 + 4x_1x_2^2}$$

Utelater ofte $\vec{a}, \vec{b}$:

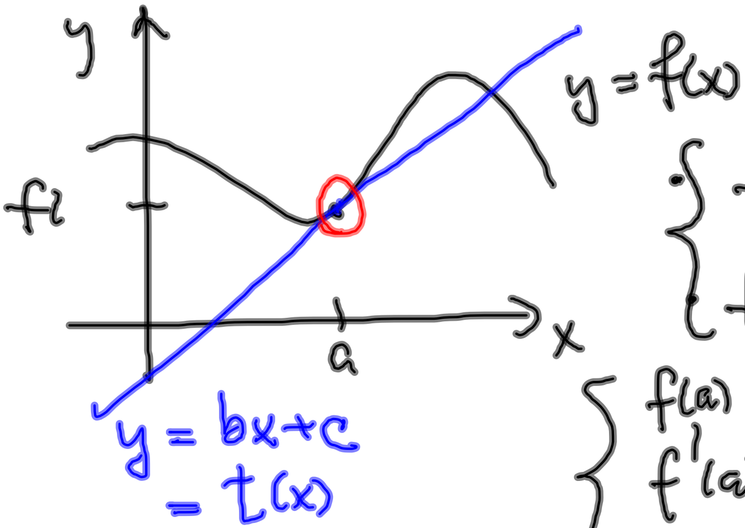
$$\frac{\partial h}{\partial x_j} = \frac{\partial f}{\partial u_1}\frac{\partial g_1}{\partial x_j} + \dots + \frac{\partial f}{\partial u_m}\frac{\partial g_m}{\partial x_j}$$

$$= \sum_{p=1}^{m} \frac{\partial f}{\partial u_p}\frac{\partial g_p}{\partial x_j}$$

$$\vec{H} = \vec{F} \circ \vec{G} \qquad \begin{aligned} \vec{H} &= (H_1, \dots, H_k) \\ \vec{F} &= (F_1, \dots, F_k) \\ \vec{G} &= (G_1, \dots, G_m) \end{aligned}$$

$$\frac{\partial H_i}{\partial x_j} = \sum_{p=1}^{m} \frac{\partial F_i}{\partial u_p} \cdot \frac{\partial G_p}{\partial x_j}$$

§2.8 LINEARISERING



$$\begin{cases} f(a) = t(a) \\ f'(a) = t'(a) \end{cases}$$

$$\begin{cases} f(a) = ba + c \\ f'(a) = b \end{cases}$$

$$\begin{aligned} t(x) &= bx + c \\ &= f'(a)x + (f(a) - f'(a)a) \\ &= f(a) + f'(a)(x-a) \end{aligned}$$

$x = a + r$

$$t(a+r) = f(a) + f'(a)r$$

La $\vec{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ være deriverbar i $\vec{a} \in \mathbb{R}^n$

Vil finne en affin
$\vec{T} : \mathbb{R}^n \to \mathbb{R}^m$
med $\vec{T}(\vec{a}) = \vec{F}(\vec{a})$ $\leftarrow$

og $\vec{T}'(\vec{a}) = \vec{F}'(\vec{a})$. $\leftarrow$

Skriver $\vec{T}(\vec{x}) = B\vec{x} + \vec{c}$ $\leftarrow$ i $\mathbb{R}^m$

B $m \times n$ matrise

$\vec{T}(\vec{a}) = B\vec{a} + \vec{c}$ $\leftarrow$

$\boxed{\underline{\text{Lemma}}\ \vec{T}'(\vec{a}) = B}$ $\swarrow$

$$\vec{T}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_{11}x_1 + \dots + b_{1n}x_n + c_1 \\ \vdots \\ b_{m1}x_1 + \dots + b_{mn}x_n + c_m \end{pmatrix} \begin{matrix} \\ i \\ \\ \end{matrix}$$

så
$$\frac{\partial T_i}{\partial \vec{x}_j} = \frac{\partial(b_{i1}x_1 + \dots + b_{ij}x_j + \dots + b_{in}x_n + c_i)}{\partial x_j}$$

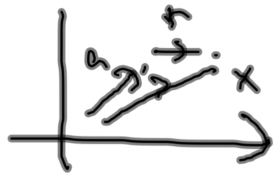
$$= b_{ij}$$

· $\vec T(\vec x) = B\vec x + \vec c$

$$\vec T(\vec a) = \vec F(\vec a) = \vec F'(\vec a)\vec a + \vec c$$

$$\vec T'(\vec a) = B = \vec F'(\vec a)$$

$$\vec c = \vec F(\vec a) - \vec F'(\vec a)\vec a$$



$$\boxed{\vec T(\vec x) = \vec F'(\vec a)\vec x + \vec F(\vec a) - \vec F'(\vec a)\vec a = \vec F(\vec a) + \vec F'(\vec a)(\vec x - \vec a)}$$

$$\vec x = \vec a + \vec r$$

$$\vec T(\vec a + \vec r) = \boxed{\vec F(\vec a) + \vec F'(\vec a)\vec r}$$

$$\vec F(\vec a + \vec r) = \boxed{\vec F(\vec a) + \vec F'(\vec a)\vec r} + \vec \sigma(\vec r)$$

deriverbar i $\vec a$ hvis $\dfrac{|\vec\sigma(\vec r)|}{|\vec r|} \to 0$ når $\vec r \to \vec 0$.

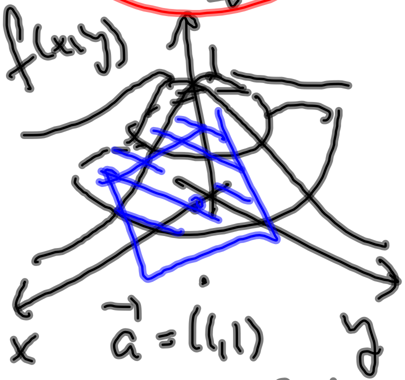
La $\boxed{T_{\vec a}\vec F = \vec T}$ Lineariseringen til $\vec F$ i $\vec a$

$$(T_{\vec a}\vec F)(\vec x) = \vec T(\vec x) = \vec F(\vec a) + \vec F'(\vec a)(\vec x - \vec a)$$

Eksempel på linearisering

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\vec{x} = (x,y) \mapsto f(\vec{x}) = f(x,y) = \frac{1}{x^2+y^2+1}$$



$$\vec{a} = (1,1)$$

$$f(\vec{a}) = f(1,1) = \frac{1}{1^2+1^2+1} = \frac{1}{3}$$

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2+y^2+1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{(x^2+y^2+1)}$$

$$\nabla f(\vec{a}) = \left(\frac{-2\cdot 1}{(1^2+1^2+1)^2}, \frac{-2}{9}\right)$$

$$\vec{x} = (x,y)$$

$$t(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

$$\shortparallel$$

$$T_{\vec{a}} f(\vec{x}) = \frac{1}{3} + \left(-\frac{2}{9}, -\frac{2}{9}\right)(x-1, y-1)$$

$$= \frac{1}{3} - \left(\frac{2}{9}\right)(x-1) - \left(\frac{2}{9}\right)(y-1)$$

$$\boxed{= -\frac{2}{9}x - \frac{2}{9}y + \frac{7}{9}}$$

TEOREM:

LA $\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ VÆRE DERIVERBAR I $\vec{a}$

OG LA

$$(T_{\vec{a}}\vec{F})(\vec{x}) = \vec{F}(\vec{a}) + \vec{F}'(\vec{a})(\vec{x}-\vec{a})$$

VÆRE LINEARISERINGEN TIL $\vec{F}$ i $\vec{a}$.

DA ER

$$\lim_{\vec{x}\to\vec{a}} \frac{\vec{F}(\vec{x}) - (T_{\vec{a}}\vec{F})(\vec{x})}{|\vec{x}-\vec{a}|} = \vec{0}$$

$$\text{"} \quad \lim_{\vec{r}\to\vec{0}} \frac{\vec{F}(\vec{a}+\vec{r}) - T_{\vec{a}}\vec{F}(\vec{a}+\vec{r})}{|\vec{r}|}$$

OG $T_{\vec{a}}\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ ER DEN ENESTE AFFINE AVBILDNINGEN MED DENNE EGENSKAPEN.

Bevis:

$$\vec{F}(\vec{a}+\vec{r}) = (T_{\vec{a}}\vec{F})(\vec{a}+\vec{r}) + \vec{\sigma}(\vec{r})$$

der $\dfrac{|\vec{\sigma}(\vec{r})|}{|\vec{r}|} \to 0$ når $\vec{r} \to \vec{0}$

$\Longleftrightarrow$ $\dfrac{\vec{\sigma}(\vec{r})}{|\vec{r}|} \to \vec{0}$ når $\vec{r} \to \vec{0}$