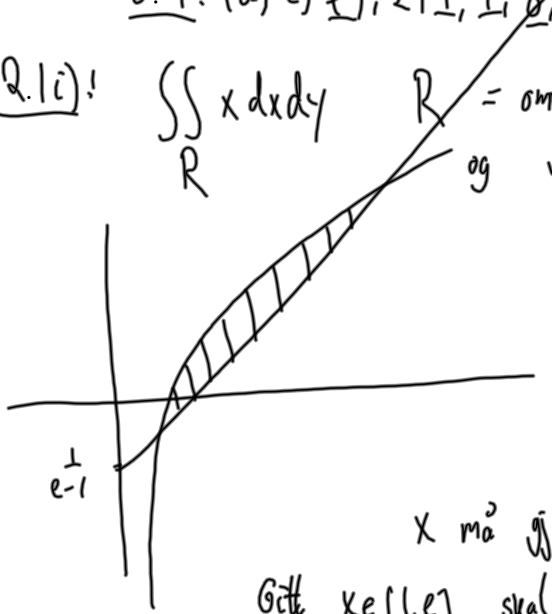
Oppgaver: 6.2: 1a) b) c) e) g) i), 2 a) c), 3

6.3: 1a) b) e) g), 3, 4

6.4: 1a) c) f), 2, 4, 7, 8, 9, 12, 17

6.2.1i): $\iint_R x\,dx\,dy$ R = området mellom $y = \ln x$ og $y = \frac{x-1}{e-1}$



Finn x-verdier hvor kurvene krysser:

$x = 1,\ x = e$

x må gjennomløpe $[1, e]$

Gitt $x \in [1,e]$, skal y gjennomløpe $\left[\frac{x-1}{e-1}, \ln x\right]$

$$\iint_R x\,dx\,dy = \int_1^e \int_{\frac{x-1}{e-1}}^{\ln x} x\,dy\,dx = \int_1^e x\left(\ln x - \frac{x-1}{e-1}\right)dx = \underbrace{\int_1^e x\ln x\,dx}_{i)} - \underbrace{\int_1^e x\frac{x-1}{e-1}\,dx}_{ii)}$$

i) $\int_1^e x\ln x\,dx = \frac{x^2}{2}\ln x\Big|_{x=1}^{x=e} - \frac{1}{2}\int_1^e x\,dx = \frac{e^2}{2}\ln e - \frac{1}{2}\ln 1 - \frac{1}{4}(e^2-1) = \frac{e^2}{2} - \frac{1}{4}(e^2-1)$

$u = \ln x,\ v' = x$

$u' = \frac{1}{x},\ v = \frac{1}{2}x^2$

ii) $\frac{1}{e-1}\int_1^e x^2 - x\,dx = \frac{1}{e-1}\left(\frac{1}{3}[e^3-1] - \frac{1}{2}(e^2-1)\right)$

$$\iint_R x\,dx\,dy = i) - ii) = \frac{e^2}{2} - \frac{1}{4}(e^2-1) - \frac{1}{e-1}\left(\frac{1}{3}[e^3-1] - \frac{1}{2}(e^2-1)\right)$$

6.3.1.g): (se eks. 3 s. 442-443)

$\iint_R (x^2+y^2)^{3/2}\,dx\,dy$, $R = \{(x,y) : (x-1)^2 + y^2 \leq 1\}$

= sirkel m/ radius 1, senter i (1,0)

$$\iint_R f(x,y)\,dx\,dy = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta} f(r\sin\theta, r\cos\theta)\,r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta} (r^2)^{3/2}\,r\,dr\,d\theta$$

$$= \int\int^{2\cos\theta} r^4\,dr\,d\theta = \int^{\pi/2} \frac{1}{5}(2\cos\theta)^5\,d\theta = \frac{2^5}{5}\int^{\pi/2} \cos^5\theta\,d\theta$$

6.4.1.f)

E området over xy-planet, under $z = 4 - (x-2)^2 - (y+1)^2$

$f(x,y) = 4 - (x-2)^2 - (y+1)^2$

$V = \iint_S f(x,y)\,dx\,dy$

S = sirkelen med radius 2, senter i (2, -1)

$(x,y) \in S$, $x = 2 + r\cos\theta$, $y = -1 + r\sin\theta$, $r \in [0,2]$, $\theta \in [0, 2\pi]$

$$\iint_S f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^2 f(2+r\cos\theta, -1+r\sin\theta)\,r\,dr\,d\theta$$

$$f(2+r\cos\theta, -1+r\sin\theta) = 4 - (r\cos\theta)^2 - (r\sin\theta)^2 = 4 - r^2$$

$$\iint_S f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left[\int_0^2 4r\,dr - \int_0^2 r^3\,dr\right]d\theta$$

$$= \int_0^{2\pi} 2r^2\Big|_{r=0}^{r=2} - \frac{1}{4}r^4\Big|_{r=0}^{r=2}\,d\theta = \int_0^{2\pi} 2\cdot 2^2 - \frac{1}{4}2^4\,d\theta = 2\pi\,[8-4] = 8\pi$$

6.4.4: Overflateintegralet til en kule med radius R.

$\vec{r}(u,v)$ parametrisering av en flate, så er overflatearealet gitt ved

$$\iint_R \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|$$

Har at kula med radius R er parametrisert ved

$$\vec{r}(u,v) = R\sin(u)\cos(v)\,\vec{i} + R\sin(u)\sin(v)\,\vec{j} + R\cos(u)\,\vec{k}, \quad 0 \le u \le \pi, \quad 0 \le v \le 2\pi$$

$$\frac{\partial \vec{r}}{\partial u} = R\cos u\cos v\,\vec{i} + R\cos u\sin v\,\vec{j} + (-R\sin u)\,\vec{k}$$

$$\frac{\partial \vec{r}}{\partial v} = -R\sin u\sin v\,\vec{i} + R\sin u\cos v\,\vec{j} + 0\,\vec{k}$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = -R^2\sin^2 u\cos v\,\vec{i} - (-R^2\sin^2 u\sin v)\,\vec{j} + R^2\underbrace{(\cos u\cos v\sin u\cos v + \cos u\sin v\sin u\sin v)}_{\cos u\sin u\cos^2 v + \cos u\sin u\sin^2 v \,=\, \cos u\sin u}\vec{k}$$

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \left(R^4\sin^4 u\cos^2 v + R^4\sin^4 u\sin^2 v + R^4\cos^2 u\sin^2 u\right)^{\frac{1}{2}}$$

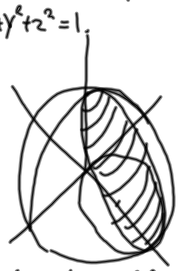
$$= R^2\left(\sin^4 u + \cos^2 u\sin^2 u\right)^{1/2} = R^2\left(\sin^2 u\,\underbrace{(\sin^2 u + \cos^2 u)}_{=1}\right)^{1/2} = R^2(\sin^2 u)^{\frac{1}{2}} = R^2|\sin u|$$

$$\iint_R \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv = \int_0^{2\pi}\int_0^{\pi} R^2|\sin u|\,du\,dv = R^2 2\pi \int_0^{\pi}\sin u\,du$$

$$= R^2 2\pi\left(-\cos u\Big|_{u=0}^{u=\pi}\right) = R^2 2\pi\, 2 = 4\pi R^2$$

6.4.7. Arealet til den delen av kuleflaten som ligger over $(x-\frac{1}{2})^2 + y^2 \leq \frac{1}{4}$

$x^2+y^2+z^2=1$.



$f(x,y) = \sqrt{1-x^2-y^2}$,

f tegner den øvre delen av kulen hvor (x,y) ligger i enhetssirkelen i xy-planet.

Overflatearealet $= \iint_S \left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2} dx\,dy$, hvor $S = \{(x,y) : (x-\frac{1}{2})^2 + y^2 \leq \frac{1}{4}\}$

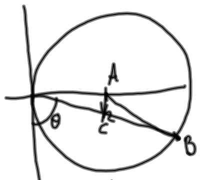
$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2\sqrt{1-x^2-y^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2-y^2}}$, $\frac{\partial f}{\partial y}(x,y) = \frac{-y}{\sqrt{1-x^2-y^2}}$

$1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2 = 1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} = \frac{1-x^2-y^2+x^2+y^2}{1-x^2-y^2} = \frac{1}{1-x^2-y^2}$

Må finne

$$\iint_S \frac{1}{\sqrt{1-x^2-y^2}} dx\,dy$$

Vil bytte til polarkoordinater

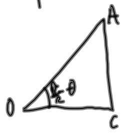


$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$B(r,\theta)$.

$r = |OB|$, C ligger OB slik at $|OC| = \frac{1}{2}|OB|$

$|OC| = \cos(\frac{\pi}{2}-\theta)\,|OA| = \frac{1}{2}\cos(\frac{\pi}{2}-\theta) = \frac{1}{2}\sin(\theta)$



Da må

$r = |OB| = 2|OC| = \frac{2}{2}\sin\theta = \sin\theta$

$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $r \in [0, \sin\theta]$

$$\iint_S \frac{1}{\sqrt{1-x^2-y^2}} dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^{\sin\theta} \frac{1}{\sqrt{1-r^2}} \cdot r\,dr\,d\theta =$$

Begynner å se på $\int_0^{\sin\theta} \frac{r}{\sqrt{1-r^2}} dr = \int_1^{1-\sin^2\theta} -\frac{1}{2}\frac{1}{u^{1/2}} du = \frac{1}{2}\int_{1-\sin^2\theta}^{1} u^{-\frac{1}{2}} du$

$u = 1-r^2$, $du = -2r\,dr$

$= u^{\frac{1}{2}}\Big|_{u=1-\sin^2\theta}^{u=1} = 1-(1-\sin^2\theta)^{\frac{1}{2}} = 1-\cos\theta$

Da blir

$$\int_{-\pi/2}^{\pi/2}\int_0^{\sin\theta} \frac{r}{\sqrt{1-r^2}} dr\,d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} (1-\cos\theta)\,d\theta = \pi - 2.$$

6.4.8: $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + f(r,\theta)\,\vec{k}$, $(r,\theta) \in A$.

Arealet til flaten er gitt ved $\iint_A \sqrt{1+\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}\, r\,dr\,d\theta$

Bevis:

Vet at

$$\iint \left|\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta}\right| dr\,d\theta = \text{arealet til flaten}$$

Har at

$\frac{\partial \vec{r}}{\partial r}(r,\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \frac{\partial f}{\partial r}\,\vec{k}$

$\frac{\partial \vec{r}}{\partial \theta}(r,\theta) = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j} + \frac{\partial f}{\partial \theta}\,\vec{k}$

$$\frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta} = \begin{vmatrix} \vec i & \vec j & \vec k \\ \cos\theta & \sin\theta & \frac{\partial f}{\partial r} \\ -r\sin\theta & r\cos\theta & \frac{\partial f}{\partial \theta} \end{vmatrix} = \vec i \left(\sin\theta \frac{\partial f}{\partial \theta} - r\cos\theta \frac{\partial f}{\partial r}\right) - \vec j \left(\cos\theta \frac{\partial f}{\partial \theta} + r\sin\theta \frac{\partial f}{\partial r}\right) + \vec k \left(\underbrace{r\cos^2\theta + r\sin^2\theta}_{= r} \right)$$

$$\left|\frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta}\right|$$

$$= \sqrt{\left(\sin\theta \frac{\partial f}{\partial \theta} - r\cos\theta \frac{\partial f}{\partial r}\right)^2 + \left(\cos\theta \frac{\partial f}{\partial \theta} + r\sin\theta \frac{\partial f}{\partial r}\right)^2 + r^2}$$

$$= \sqrt{\sin^2\theta \frac{\partial f}{\partial \theta}^2 - 2\sin\theta \frac{\partial f}{\partial \theta} r\cos\theta \frac{\partial f}{\partial r} + r^2\cos^2\theta \frac{\partial f}{\partial r}^2 + \cos^2\theta \frac{\partial f}{\partial \theta}^2 + 2\cos\theta \frac{\partial f}{\partial \theta} r\sin\theta \frac{\partial f}{\partial r} + r^2\sin^2\theta \frac{\partial f}{\partial r}^2 + r^2}$$

$$= \sqrt{\frac{\partial f}{\partial \theta}^2 + r^2\left(\frac{\partial f}{\partial r}\right)^2 + r^2} = r\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}$$

Altså blir

$$\iint_A \left|\frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta}\right| dr\,d\theta = \iint_A r\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}\, dr\,d\theta$$

6.4.9. Regn ut flateintegralet $\iint_T x^2\,dS$, T gitt ved $z = x^2 + y^2$, $x^2 + y^2 \le 1$.

T parametrisert med polarkoordinater

$x = r\cos\theta$, $y = r\sin\theta$, $r \in [0,1]$, $\theta \in [0, 2\pi]$

$z = x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

Da blir flateintegralet gitt ved

$$\iint_T f(x,y)\,dS = \iint_A f(r\cos\theta, r\sin\theta) \left|\frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta}\right| dr\,d\theta$$ def. 6.4.1 s. 451.

$$\vec r(r,\theta) = r\cos\theta\,\vec i + r\sin\theta\,\vec j + r^2\,\vec k$$

$$\left|\frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta}\right| = r\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2} = r\sqrt{1+4r^2}$$

fra forrige oppgave, hvis $f(r,\theta) = r^2$; $\frac{\partial f}{\partial r} = 2r$, $\frac{\partial f}{\partial \theta} = 0$

Da blir

$$\iint_T x^2\,dS = \iint_A r^2\cos^2\theta \cdot r\sqrt{1+4r^2}\, dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3\sqrt{1+4r^2}\cos^2\theta\, dr\,d\theta$$

$$* \int_0^1 r^3\sqrt{1+4r^2}\,dr = \frac{2}{3\cdot 8}(1+4r^2)^{3/2} r^2 \Big|_{r=0}^{r=1} - \frac{2\cdot 2}{3\cdot 8}\int_0^1 r(1+4r^2)^{3/2}\,dr$$

$u = 1+4r^2$, $du = 8r\,dr$

$u = r^2$, $v' = r\sqrt{1+4r^2}$

$u' = 2r$, $v = \frac{2\cdot 1}{3\cdot 8}(1+4r^2)^{3/2}$

$$= \frac{2\cdot 5^{3/2}}{3\cdot 8} - \frac{2\cdot 2}{3\cdot 8}\frac{1}{8}\int_0^5 u^{3/2}\,du$$

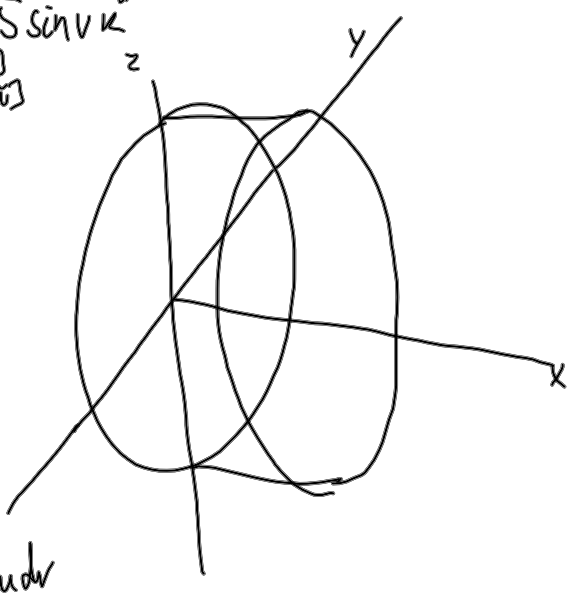
$$= \frac{2\cdot 5^{3/2}}{3\cdot 8} - \frac{2\cdot 2}{3\cdot 8\cdot 8}\cdot\frac{2}{5}\cdot 5^{5/2}$$

6.4.12: Sylinderflaten T har parametrisering

$$\vec{r}(u,v) = u\vec{i} + \sqrt{5}\cos v\,\vec{j} + \sqrt{5}\sin v\,\vec{k}$$

$u \in [0,2]$
$v \in [0, 2\pi]$

Flaten er en sylinder som ligger langs x-aksen og har radius $\sqrt{5}$



Finn flateintegralet

$$\iint_T x\,dS = \int_0^{2\pi}\int_0^{2} u \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv$$

Nå er
$$\frac{\partial \vec{r}}{\partial u} = \vec{i} + 0\cdot\vec{j} + 0\cdot\vec{k}$$
$$\frac{\partial \vec{r}}{\partial v} = 0\cdot\vec{i} - \sqrt{5}\sin v\,\vec{j} + \sqrt{5}\cos v\,\vec{k}$$

$$\Rightarrow \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = 0\cdot\vec{i} - (\sqrt{5}\cos v)\,\vec{j} + (-\sqrt{5}\sin v)\,\vec{k}$$
$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{\sqrt{5}^2\cos^2 v + \sqrt{5}^2\sin^2 v} = \sqrt{5}$$

Da blir

$$\iint_T x\,dS = \int_0^{2\pi}\int_0^{2} u \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv = \sqrt{5}\int_0^{2\pi}\int_0^{2} u^2\,du\,dv = \frac{2\pi\cdot\sqrt{5}\cdot 2^2}{2} = \underline{\underline{2\sqrt{5}\pi}}$$