6.5: 1a) b) d), 3, 5, 7, 8, 10, 13

6.7: 1a) b), 3a) c), 5 a) b) c), 7

6.8: 1, 2, 3, 6

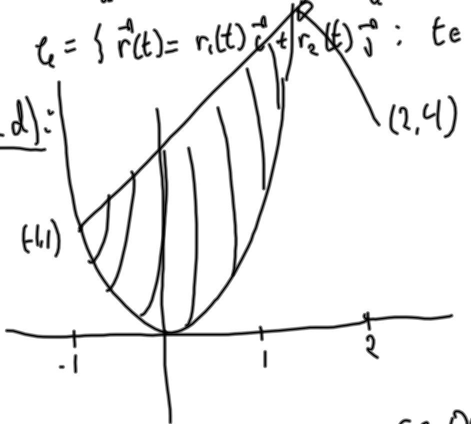
Greens teorem: $\mathcal{C}$ lukket kurve

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$ R = området omsluttet av $\mathcal{C}$.

$$\int_{\mathcal{C}} P\,dx = \int_a^b P(\vec{r}(t))\, r_1'(t)\,dt, \quad \int_{\mathcal{C}} Q\,dy = \int_a^b Q(\vec{r}(t))\, r_2'(t)\,dt$$

hvor $\mathcal{C} = \{\vec{r}(t) = r_1(t)\vec{i} + r_2(t)\vec{j} : t \in [a,b]\}$

6.5.1 d):



$$\int_{\mathcal{C}} \underbrace{(x^2y + xe^x)}_{P}\,dx + \underbrace{(xy^3 + e^{\sin y})}_{Q}\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$Q = xy^3 + e^{\sin y}, \quad \frac{\partial Q}{\partial x} = y^3, \quad P = x^2y + xe^x, \quad \frac{\partial P}{\partial y} = x^2$$

Likning for linja: $\frac{y - y_0}{x - x_0} = \frac{\Delta y}{\Delta x} = \frac{4-1}{2-(-1)} = \frac{3}{3} = 1.$

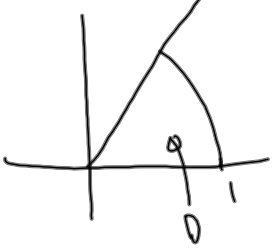
$$\frac{y-1}{x-(-1)} = 1. \quad y - 1 = x + 1, \quad y = x + 2.$$

$R = \{(x,y) : x \in [-1,2],\ y \in [x^2, x+2]\}$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{-1}^{2}\int_{x^2}^{x+2} (y^3 - x^2)\,dy\,dx = \int_{-1}^{2} \frac{1}{4} y^4 \Big|_{y=x^2}^{y=x+2} - x^2(x+2-x^2)\,dx$$

$$= \int_{-1}^{2} \frac{1}{4}\left((x+2)^4 - x^8\right) - x^3 - 2x^2 + x^4\,dx$$

6.5.10: $D = \{(x,y) ; x^2+y^2 \le 1,\ x,y \ge 0,\ y \le x\}$

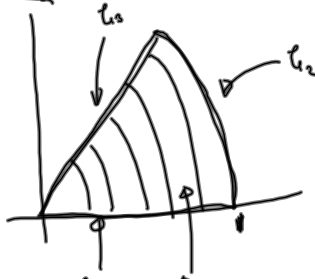


$$I = \iint_D (x + y^2)\,dx\,dy = \int_0^{\pi/4}\int_0^1 (r\cos\theta + r^2\sin^2\theta)\, r\,dr\,d\theta$$

$\left(D = \{(x,y) : x = r\cos\theta,\ y = r\sin\theta,\ r \in [0,1],\ \theta \in [0, \tfrac{\pi}{4}]\}\right)$

$$= \int_0^{\pi/4} \left(\frac{1}{3}\cos\theta + \frac{1}{4}\sin^2\theta\right) d\theta$$

6.5.10 b): $\mathcal{C}$ omkretsen til D.



$$\int_{\mathcal{C}} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3}$$

$\mathcal{C}_1$ D

$$\mathcal{C}_1 = \left\{ \tfrac{\sqrt{2}}{2}(\vec{i}+\vec{j}) - t(\vec{i}+\vec{j}) \ :\ t \in [0,1] \right\}, \quad \mathcal{C}_2 = \left\{ \cos t\,\vec{i} + \sin t\,\vec{j} \ :\ t \in [0, \tfrac{\pi}{4}] \right\}$$

$$\mathcal{C}_3 = \{ \qquad : t \in [0$$

$$\iint_D (x+y^2)\,dx\,dy = \int_{\mathcal{C}} P\,dx + Q\,dy, \quad \text{dersom } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x+y^2$$

$$P = 0, \quad \frac{\partial Q}{\partial x} = x + y^2. \quad Q = \tfrac{1}{2}x^2 + xy^2$$

$$\iint_D (x+y^2)\,dx\,dy = \int_{\mathcal{C}} \left(\tfrac{1}{2}x^2 + xy^2\right) dy$$

$$= \int_{\mathcal{C}_1} \tfrac{1}{2}x^2 + xy^2\,dy + \int_{\mathcal{C}_2} \tfrac{1}{2}x^2 + xy^2\,dy + \int_{\mathcal{C}_3} \tfrac{1}{2}x^2 + xy^2\,dy$$

$$\int_{\mathcal{C}} Q\,dy = \int_a^b Q(\vec{r}(t))\, r_2'(t)\,dt.$$

$$\int_{\mathcal{C}_1} \tfrac{1}{2}x^2 + xy^2\,dy = \int_0^1 \left(\tfrac{1}{2}t^2 + t\cdot 0^2\right)\cdot 0\,dt = 0$$

$$\int_{\mathcal{C}_2} \tfrac{1}{2}x^2 + xy^2\,dy = \int_0^{\pi/4} \left(\tfrac{1}{2}\cos^2 t + \cos t \sin^2 t\right)\cos t\,dt$$

$$\int_{\mathcal{C}_3} \tfrac{1}{2}x^2 + xy^2\,dy = \int_0^1 \left(\tfrac{1}{2}(1-t)^2 + (1-t)(1-t)^2\right)(-1)\,dt$$

6.5.13: $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$, konservativt felt, $\mathcal{C}$ lukket kurve

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = 0.$$

Bevis ved Greens teorem:

Siden $\vec{F}$ konservativ, så fins en funksjon $\phi : \mathbb{R}^2 \to \mathbb{R}$ slik at $\vec{F} = \nabla\phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j}$

$P = \frac{\partial \phi}{\partial x}$, $Q = \frac{\partial \phi}{\partial y}$. $\quad \vec{r}(t) = r_1(t)\vec{i} + r_2(t)\vec{j}$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_a^b P(\vec{r}(t))\,r_1'(t)\,dt + \int_a^b Q(\vec{r}(t))\,r_2'(t)\,dt$$

$$= \int_{\mathcal{C}} P\,dx + Q\,dy \underset{\substack{\uparrow \\ \text{Greens} \\ \text{teorem}}}{=} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R \underbrace{\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)}_{=0} dx\,dy = 0 \quad \blacksquare$$

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$$

Skifte av variabel i dobbeltintegral

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

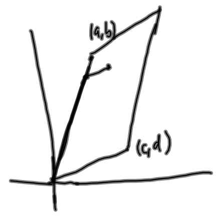
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Merk: Kan være enklere å beregne

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$, da er $\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$

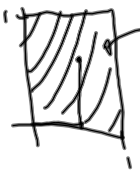
6.7.7: a) A = parallellogrammet utspent av $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$

$= \left\{ \lambda_1 \begin{pmatrix} a \\ b \end{pmatrix} + \lambda_2 \begin{pmatrix} c \\ d \end{pmatrix} : \lambda_1, \lambda_2 \in [0,1] \right\}$



Avbildningen $\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix} = M\begin{pmatrix} u \\ v \end{pmatrix}$, hvor $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

avbilder $K = [0,1]^2$ på A



$K = \{ \lambda_1 \vec{e}_1 + \lambda_2 \vec{e}_2 : \lambda_1, \lambda_2 \in [0,1] \}$

Må vise at for $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in A$, så fins $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in K$

slik at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = T\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = M\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

Siden $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in A$, fins λ_1, λ_2 slik at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda_1 \begin{pmatrix} a \\ b \end{pmatrix} + \lambda_2 \begin{pmatrix} c \\ d \end{pmatrix}$

Definér: $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \lambda_1 \vec{e}_1 + \lambda_2 \vec{e}_2$.

Da blir $M\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = M(\lambda_1 \vec{e}_1 + \lambda_2 \vec{e}_2) = \lambda_1 \underbrace{M\vec{e}_1}_{\begin{pmatrix} a \\ b \end{pmatrix}} + \lambda_2 \underbrace{M\vec{e}_2}_{\begin{pmatrix} c \\ d \end{pmatrix}} = \lambda_1 \begin{pmatrix} a \\ b \end{pmatrix} + \lambda_2 \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

b) $\iint_A f(x,y)\,dx\,dy = |\det M| \int_0^1\int_0^1 f(au+cv, bu+dv)\,du\,dv$

$\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au+cv \\ bu+dv \end{pmatrix}$. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \det M$

$x(u,v) = au+cv$

$\frac{\partial x}{\partial u} = a, \ \frac{\partial x}{\partial v} = c.$

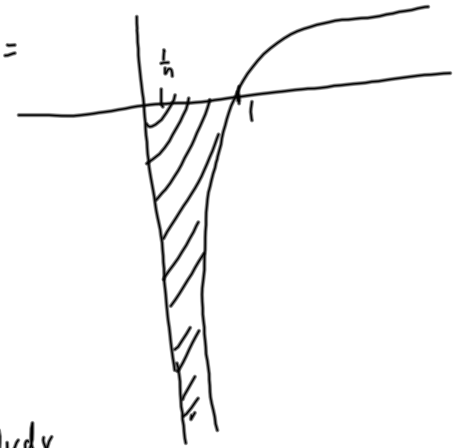
Da blir

$$\iint_A f(x,y)\,dx\,dy \underset{\substack{\text{skift} \\ \text{variabel}}}{=} \iint_K f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = |\det M| \int_0^1\int_0^1 f(au+cv, bu+dv)\,du\,dv$$

c) $\iint_A e^{2x-3y}\,dx\,dy = |\det M| \int_0^1\int_0^1 e^{2(au+cv)-3(bu+dv)}\,du\,dv = 7\int_0^1\int_0^1 e^{7u} e^{-7v}\,du\,dv = \frac{1}{7}(e^7-1)(1-e^{-7})$

A utspent av $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $M = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$, $\det M = 6-(-1) = 7$

6.8.3: Konvergerer $\iint_A x\,dx\,dy$ når $A =$



$A = \{(x,y) : 0 \le x \le 1,\ \ln x \le y \le 0\}$

La

$A_n = \{(x,y) : \frac{1}{n} \le x \le 1,\ \ln x \le y \le 0\}$

Da blir

$\iint_A x\,dx\,dy = \lim_{n\to\infty} \iint_{A_n} x\,dx\,dy = \lim_{n\to\infty} \int_{\frac{1}{n}}^{1}\int_{\ln x}^{0} x\,dy\,dx$

$= \lim_{n\to\infty} \int_{\frac{1}{n}}^{1} -x\ln x\,dx \overset{\varphi}{=} \lim_{n\to\infty} \left(-\frac{1}{2}x^2 \ln x \Big|_{x=\frac{1}{n}}^{x=1} + \int_{\frac{1}{n}}^{1} \frac{1}{2}x\,dx \right)$

$\boxed{\text{delvis integrasjon: } u = \ln x,\ v' = x \quad u' = \frac{1}{x},\ v = \frac{1}{2}x^2}$

$= \lim_{n\to\infty} \Big(\underbrace{\frac{1}{2}\left(\frac{1}{n}\right)^2 \ln\left(\frac{1}{n}\right)}_{i)} + \underbrace{\int_{\frac{1}{n}}^{1} \frac{1}{2}x\,dx}_{ii} \Big)$

ii) $\lim_{n\to\infty} \int_{\frac{1}{n}}^{1} \frac{1}{2}x\,dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{4}$

i) $\lim_{n\to\infty} \frac{1}{2} \frac{\ln n^{-1}}{n^2} \overset{L'H}{=} \frac{1}{2}\lim_{n\to\infty} \frac{\frac{1}{n^{-1}}\cdot -n^{-2}}{2n} = \frac{1}{2}\lim_{n\to\infty} \frac{-\frac{n}{n^2}}{2n} = \frac{1}{2}\lim_{n\to\infty} -\frac{1}{2n^2} = 0$

Da blir

$\iint_A x\,dx\,dy = \frac{1}{4}$

6.8.6: $A = \{(x,y) : x^2+y^2 \ge 1\}$. For hvilke p konvergerer

$\iint_A \frac{1}{(x^2+y^2)^p}\,dx\,dy$

$= \lim_{n\to\infty} \iint_{A\cap B(0,n)} \frac{1}{(x^2+y^2)^p}\,dx\,dy$

$A\cap B(0,n) = \{(x,y) : x = r\cos\theta,\ y = r\sin\theta$

$\theta \in [0,2\pi],\ r \in [1,n]\}$

$p \ne 1.$

$\iint_{A\cap B(0,n)} \frac{1}{(x^2+y^2)^p}\,dx\,dy = \int_0^{2\pi}\int_1^n \frac{r}{r^{2p}}\,dr\,d\theta = \int_0^{2\pi}\int_1^n r^{1-2p}\,dr\,d\theta = 2\pi \frac{r^{2-2p}}{2-2p}\Big|_{r=1}^{r=n}$

$= \frac{2\pi}{2-2p}\left(n^{2-2p} - 1\right),$

$\lim_{n\to\infty} \iint_{A\cap B(0,n)} \frac{1}{(x^2+y^2)^p}\,dx\,dy = \frac{2\pi}{2-2p} \lim_{n\to\infty} \left(n^{2-2p} - 1\right),$

Så $2-2p < 0$ gir at integralet konvergerer, dvs $1 < p$.