Oppgaver:

3.1: 5d), 3.4: 1, 2, 3, ~~5~~, 6, 8, ~~10~~, 14

3.5: 1, 2, 4, 5, 9, 11, 12, 3.6: 2, 6, 9, 10

3.4.5: $F(x,y,z) = yz\vec{i} + x\vec{j} + xy\vec{k}$

$\vec{r}(t) = t\vec{i} + \arctan t\,\vec{j} + t\vec{k}, \quad t \in [0,1]$

Finn linjeintegralet

$$\int_C F \cdot d\vec{r} = \int_0^1 F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$\vec{r}\,'(t) = \vec{i} + \frac{1}{1+t^2}\vec{j} + \vec{k}, \quad F(\vec{r}(t)) = t\arctan t\,\vec{i} + t\vec{j} + t\arctan t\,\vec{k}$

Da blir

$$F(\vec{r}(t)) \cdot \vec{r}\,'(t) = t\,\mathrm{arct}\,t + \frac{t}{1+t^2} + t\arctan t = 2t\arctan t + \frac{t}{1+t^2}$$

Skal finne

$$\int_0^1 2t\arctan t + \frac{t}{1+t^2}\,dt = \underbrace{2\int_0^1 t\arctan t\,dt}_{i)} + \underbrace{\int_0^1 \frac{t}{1+t^2}\,dt}_{ii)}$$

$u' = 2t, \; v = \arctan t$

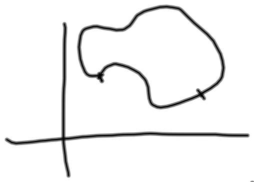
$u = t^2, \; v' = \frac{1}{1+t^2}$

$$\frac{t^2}{1+t^2} = \frac{1+t^2-1}{1+t^2} = \frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}$$

$$i) = t^2\arctan t\Big|_{t=0}^{t=1} - \int_0^1 \frac{t^2}{1+t^2}\,dt = \frac{\pi}{4} - \int_0^1 \cancel{\frac{1+t^2}{1+t^2}} - \frac{1}{1+t^2}\,dt$$

$$= \frac{\pi}{4} - 1 - \arctan t\Big|_{t=0}^{t=1}$$

3.4.12: $\mathcal{C}$ er lukket, $\vec{r}:[0,1]\to\mathbb{R}^n$. $\vec{r}(0)=\vec{r}(1)$



Vi skal vise at $\int_{\mathcal{C}} F\cdot d\vec{r}$ er det samme uansett hvor vi begynner

La $\vec{r}_0(t)=\vec{r}(t+c)$.

$$\int_{\mathcal{C}} F\cdot d\vec{r}_0 = \int_0^1 F(\vec{r}_0(t))\cdot\vec{r}_0{}'(t)\,dt = \int_0^1 F(\vec{r}(t+c))\cdot\vec{r}\,'(t+c)\,dt$$

skift variabel, $s=t+c$,

$$= \int_c^{1+c} F(\vec{r}_1(s))\cdot\vec{r}_1{}'(s)\,ds,$$ hvor $\vec{r}_1(t)=\begin{cases}\vec{r}(t) & t\in[0,1]\\ \vec{r}(t-1), & t\in[1,2]\end{cases}$

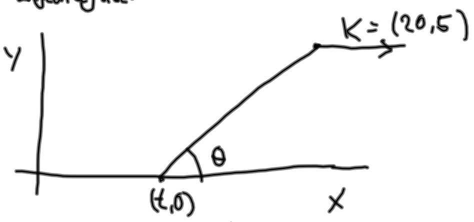
er veldefinert siden $\vec{r}(0)=\vec{r}(1)$. Men

$\vec{r}_1$ og $\vec{r}$ er ekvivalente parametriseringer av samme kurve,

$\phi:[0,1]\to[c,1+c]$, $\phi(t)=t+c$. er kont. deriverbar.

Da er linjeintegralet det samme om vi bruker $\vec{r}$ eller $\vec{r}_1$

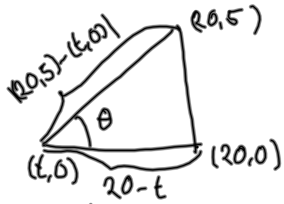
3.4.14:



Skal finne arbeidet utført på båten.

Kraften utført på båten ved tiden t som brukes en kort strekning Δt.

$|F|\Delta t\cdot\cos\theta$



$$\cos\theta = \frac{20-t}{|(20,5)-(t,0)|} = \frac{20-t}{\sqrt{(20-t)^2+25}}$$

$|F| = K$

Arbeidet utført ved tiden t for å flytte båten en Δt er lik

$$|F|\Delta t\cos\theta = K\frac{20-t}{\sqrt{(20-t)^2+25}}\,\Delta t$$

Summerer vi dette opp og lar Δt bli liten konvergerer dette mot

$$\int_0^{20}\frac{K(20-t)}{\sqrt{(20-t)^2+25}}\,dt$$

b) Finn $\int_0^{20}\frac{K(20-t)}{\sqrt{(20-t)^2+25}}\,dt$.

$$\frac{d}{dt}\sqrt{(20-t)^2+25} = \frac{1}{2}\frac{1}{\sqrt{(20-t)^2+25}}\cdot(2)(20-t)$$

$$K\int_0^{20}\frac{20-t}{\sqrt{(20-t)^2+25}}\,dt = -K\sqrt{(20-t)^2+25}\,\Big|_{t=0}^{t=20}$$

$$= -K\left(5-\sqrt{20^2+25}\right) = K\left(\sqrt{20^2+25}-5\right)$$

3.5.9.

$$F(x,y) = y^2 e^{xy^2}\vec{i} + (2xye^{xy^2}+1)\vec{j}$$

$\int_C F\cdot d\vec{r}$, $\quad C$ = sirkelen med sentrum $(1,-1)$, radius 5

$$\frac{\partial F_1}{\partial y}(x,y) = 2ye^{xy^2} + 2y^3x\,e^{xy^2}$$

$$\frac{\partial F_2}{\partial x}(x,y) = 2y\,e^{xy^2} + 2xy^3e^{xy^2}$$

Siden $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ er F konservativ, dvs $\nabla\phi = F$ for en funksjon ϕ. Da er $\int_C F\cdot d\vec{r} = \int_C \nabla\phi\cdot d\vec{r} = 0$ siden C er lukket.

3.5.11:

$$F(x,y,z) = ze^{xz+y}\vec{i} + (e^{xz+y}+2z)\vec{j} + (xe^{xz+y}+2y)\vec{k}$$

Sjekk at

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$, så F er konservativ

Siden C er en lukket kurve, er $\int_C F\cdot d\vec{r} = 0$

3.5.12:

$F(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$ - definert på $\mathbb{R}^2 \setminus \{(0,0)\}$

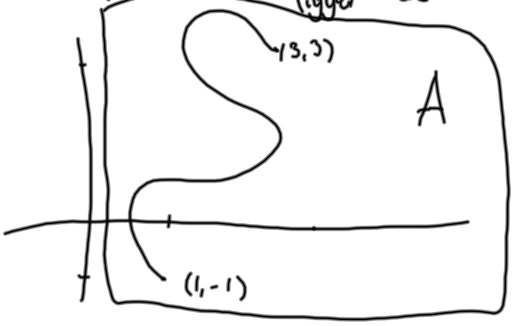
a) $\phi_1(x,y) := \arctan\frac{y}{x} + C$, $\quad \nabla\phi_1 = F$

$$\nabla\phi_1(x,y) = \left(\frac{\partial\phi_1}{\partial x}(x,y), \frac{\partial\phi_1}{\partial y}(x,y)\right)$$

$$\frac{\partial\phi_1}{\partial x}(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2}$$

og $\frac{\partial\phi_1}{\partial y}(x,y) = \frac{x}{x^2+y^2}$

b) $\int_C F\cdot d\vec{r}$, $\quad C$ = glatt kurve som starter i $(1,-1)$ og ender i $(3,3)$ ligger til høyre for y-aksen



$F = \nabla\phi_1$ på A

$$\int_C F\cdot d\vec{r} = \int_C \nabla\phi_1\cdot d\vec{r}$$

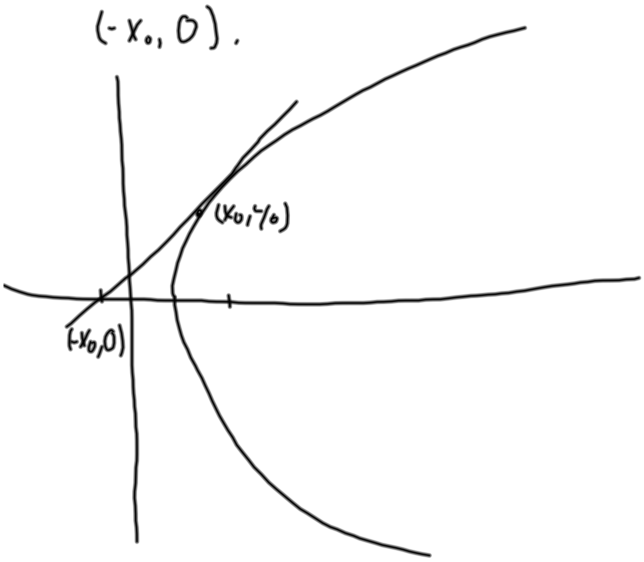
$$= \phi_1(\vec{r}(b)) - \phi_1(\vec{r}(a))$$

$$= \phi_1(3,3) - \phi_1(1,-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$\boxed{\begin{aligned}\phi_1(3,3) &= \arctan\tfrac{3}{3} = \arctan 1 = \tfrac{\pi}{4}\\ \phi_1(1,-1) &= \arctan(-1) = -\tfrac{\pi}{4}\end{aligned}}$$

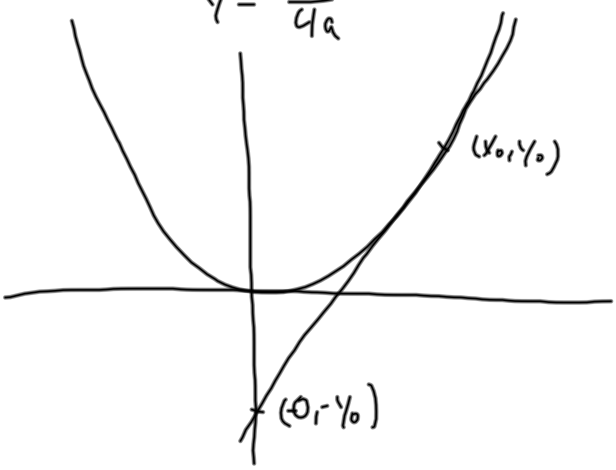
3.6.9:

Vis at tangenten til $y^2 = 4ax$ i (x_0, y_0) skjærer x-aksen i $(-x_0, 0)$.



For enkelhets skyld

$$y = \frac{x^2}{4a}$$



Likning for tangenten i punktet (x_0, y_0).

$l(x) = cx + d$, $l'(x) = c$

$l'(x_0) = \frac{x_0}{2a}$, $l(x_0) = y_0$

$f(x) = \frac{x^2}{4a}$, $f'(x) = \frac{x}{2a}$

Da må $c = \frac{x_0}{2a}$

$cx_0 + d = y_0$

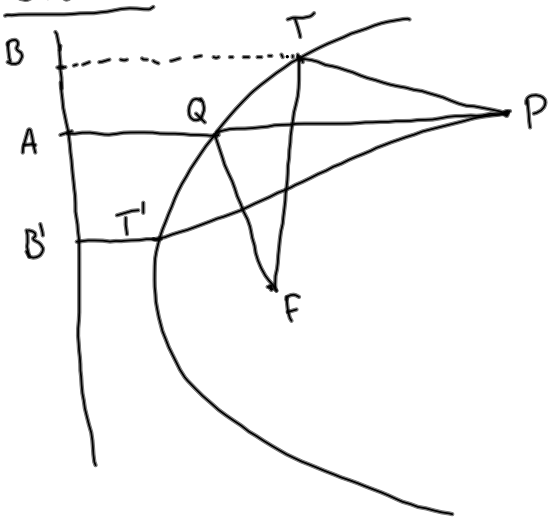
$d = y_0 - \frac{x_0^2}{2a}$.

Så $l(x) = \frac{x_0}{2a}x + y_0 - \frac{x_0^2}{2a}$,

$l(0) = y_0 - \frac{x_0^2}{2a} = -y_0$ siden (x_0, y_0) ligger på parabelen, og da er $y_0 = \frac{x_0^2}{4a}$, dvs $2y_0 = \frac{x_0^2}{2a}$

.

3.6.10:



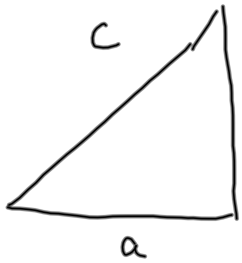
Vil vise at

$|PQ| + |QF| < |PT| + |TF|$

$|PQ| + |QF| = |PQ| + |QA|$

$= |PA| < |PT| + |TB|$

$= |PT| + |TF|$



$c^2 = b^2 + a^2$

$> a^2$

$\Rightarrow \quad c > a$