Oppgaver:

5.9: 11, 13, 15, 16, 17, 18, 19,

5.10: 1a) b) c) d), 2, 7, 8, 11, 12, 13. 14, 16, 17

12.1: 1, 3, 4a) b) c), 5

5.9.18: To bedrifter

A produserer x enheter per måned

B —"— y —"—

Fortjeneste til A: $P(x,y) = 12000x - \frac{x^2}{2} - \frac{y^2}{4}$

—"— B: $Q(x,y) = 12000y - \frac{y^2}{2} - \frac{x^2}{6}$

a) Hver bedrift vil maksimere fortjeneste, dvs

$\max_x P(x,y)$, $\max_y Q(x,y)$

Deriver $\frac{\partial P}{\partial x} = 0$ for bedrift A, $\frac{\partial P}{\partial x} = 12000 - x$

$\frac{\partial Q}{\partial y} = 0$ for bedrift B, $\frac{\partial Q}{\partial y} = 12000 - y$

$P(\underbrace{12000}_{= 12\cdot 10^3}, 12000) = 12\cdot 10^3 \cdot 12\cdot 10^3 - \frac{(12\cdot 10^3)^2}{2} - \frac{(12\cdot 10^3)^2}{4} = \frac{(12\cdot 10^3)^2}{4} = \frac{12}{4}\, 12\cdot 10^6 = 36\cdot 10^6$

$Q(12000, 12000) = 48\cdot 10^6$

b) Sammarbeid om størst fortjeneste

maksimér $(P+Q)(x,y) = P(x,y) + Q(x,y)$

Stasjonære punkter $\frac{\partial}{\partial x}(P+Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} = 12000 - x - \frac{x}{3} = 12000 - \frac{4x}{3} = 0$, $x = 9000$

$\frac{\partial}{\partial y}(P+Q) = \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y} = -\frac{y}{2} + 12000 - y = 12000 - \frac{3y}{2} = 0$, $y = 8000$

$P(9000, 8000) = 51.5\cdot 10^6$, $Q(9000, 8000) = 50.5\cdot 10^6$

c) A maksimerer $P+Q$. $\frac{\partial}{\partial x}(P+Q) = 12000 - \frac{4x}{3} = 0 \Rightarrow x = 9000$

B maksimerer Q $\frac{\partial}{\partial y} Q(y) = 0$, $y = 12\,000$

$P(9000, 12000) = 31.5\cdot 10^6$, $Q(9000, 12000) = 58.5\cdot 10^6$

5.9.19: $f(x,y) = x^4 + y^4$

a) (0,0) stasjonært punkt: $\frac{\partial f}{\partial x} = 4x^3$, $\frac{\partial f}{\partial y} = 4y^3$, så (0,0) er et stasjonært punkt.

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(0,0) er et minimumspunkt: $f(x,y) = x^4 + y^4 \geq 0$, $f(0,0) = 0^4 + 0^4 = 0$

$\Rightarrow$ (0,0) minimumspunkt.

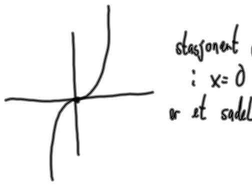
b) $g(x,y) = -f(x,y) = -x^4 - y^4$. $Hg(x,y) = \begin{pmatrix} -12x^2 & 0 \\ 0 & -12y^2 \end{pmatrix}$

(0,0) er maxspunkt siden $g(x,y) \leq 0$, $g(0,0) = 0$.

c) $h(x,y) = x^3 + y^3$. $\frac{\partial h}{\partial x} = 3x^2$, $\frac{\partial h}{\partial y} = 3y^2$. (0,0) er et stasjonært punkt

$Hh(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$. $Hh(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(0,0) er et sadelpunkt ($f(x) = x^3$



stasjonært punkt: $x = 0$, som er et sadelpunkt)

5.10.8: $f(x,y) = \ln(x^2+y^2+1) - \frac{x^2}{2} + y^2$

a) $\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2+1} \cdot 2x - x$, $\frac{\partial f}{\partial y} = \frac{1}{x^2+y^2+1} \cdot 2y + 2y$

$\frac{2x}{x^2+y^2+1} - x = 0$, $\frac{2y}{x^2+y^2+1} + 2y = 0$, (0,0) er et ~~så~~ stasjonært punkt.

Fra $\frac{2y}{x^2+y^2+1} + 2y = 0$ Anta $y \neq 0$

$\frac{2}{x^2+y^2+1} + 2 = 0$, $2 = -2(x^2+y^2+1)$, umulig.

Så $y = 0$.

Fra $\frac{2x}{x^2+y^2+1} - x = 0$. Anta $x \neq 0$

$\frac{2}{x^2+1} - 1 = 0$. $2 = (x^2+1) \Rightarrow x^2 = 1$, $x = \pm 1$.

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \frac{2(x^2+y^2+1) - 4x^2}{(x^2+y^2+1)^2} - 1 & \frac{-4xy}{(x^2+y^2+1)^2} \\ \frac{-4xy}{(x^2+y^2+1)^2} & \frac{2(x^2+y^2+1) - 4y^2}{(x^2+y^2+1)^2} + 2 \end{pmatrix}$$

$Hf(0,0) = \begin{pmatrix} \frac{2 \cdot 1 - 0}{1} - 1 & 0 \\ 0 & \frac{2 \cdot 1 - 0}{1} + 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. $\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} = 4 > 0$, $1 > 0 \Rightarrow$ minimumspunkt

$Hf(\pm 1,0) = \begin{pmatrix} \frac{2 \cdot 2 - 4}{(2)^2} - 1 & 0 \\ 0 & \frac{2 \cdot 2 - 0}{(2)^2} + 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$. $\begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} = -3 < 0 \Rightarrow$ sadelpunkt

b) minimér/maksimér $f(x,y)$ på $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

$\max_{x^2+y^2 \leq 1} f(x,y) = \max_{x^2+y^2=1} f(x,y)$ siden f ikke har stasjonære punkter på $x^2+y^2 < 1$ bortsett fra (0,0).

$\max_{g(x,y)=1} f(x,y)$. Bytt til polarkoordinater. $x = r\cos\theta$, $y = r\sin\theta$

$\max_{g(r,\theta)=1} f(r,\theta)$. $f(r,\theta) = \ln(r^2+1) - \frac{r^2\cos^2\theta}{2} + r^2\sin^2\theta$

$g(r,\theta) = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2 = 1$. Da må $r = 1$.

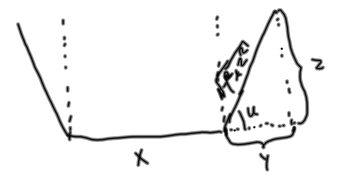
Står igjen med $f(\theta) = f(1,\theta) = \ln(2) - \frac{\cos^2\theta}{2} + \sin^2\theta$

derivér $f'(\theta) = 0$, løs for θ

$f'(\theta) = \cos\theta\sin\theta + 2\sin\theta\cos\theta = \frac{3}{2}\sin 2\theta = 0$, $2\theta = 0$, $2\theta = \pi$, $2\theta = 2\pi$, $2\theta = 3\pi$
$\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$, $\theta = \frac{3\pi}{2}$

Sett inn, og sjekk!

5.10.13:



$x + 2\sqrt{y^2+z^2} = b$

Areal $= xz + 2\frac{yz}{2} = xz + yz$

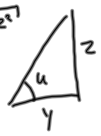
$\underset{x+2\sqrt{y^2+z^2}=b}{\text{maks}}\ xz+yz$ $\qquad f(x,y,z) = xz+yz$

$\qquad g(x,y,z) = x + 2\sqrt{y^2+z^2}$

$\frac{\partial f}{\partial x} = z \qquad \frac{\partial g}{\partial x} = 1 \qquad z = \lambda$

$\frac{\partial f}{\partial y} = z \qquad \frac{\partial g}{\partial y} = \frac{2y}{\sqrt{y^2+z^2}} \qquad z = \lambda \frac{2y}{\sqrt{y^2+z^2}}$

$\frac{\partial f}{\partial z} = x+y \qquad \frac{\partial g}{\partial z} = \frac{2z}{\sqrt{y^2+z^2}} \qquad x+y = \lambda \frac{2z}{\sqrt{y^2+z^2}}$



$\Rightarrow u = \frac{\pi}{3}$

$x = 2y = \frac{b}{3}$, $z = \sqrt{3}y = \frac{\sqrt{3}b}{6}$, $\frac{z}{y} = \sqrt{3}$ $\qquad x = \frac{b}{3}$

5.10.14:

$A(x,y,z) = \sqrt{s(s-x)(s-y)(s-z)}$, hvor $s = \frac{x+y+z}{2}$

$\underset{x+y+z=2s}{\text{maks}}\ A(x,y,z)$ $\qquad$ La $f(x,y,z) = A^2(x,y,z) = s(s-x)(s-y)(s-z)$

$\shortparallel$

$\left(\underset{x+y+z=2s}{\text{maks}}\ f(x,y,z)\right)^{1/2}$ $\qquad$ (siden $0<a<b \Leftrightarrow \sqrt{a} < \sqrt{b}$)

$g(x,y,z) = x+y+z$.

$\frac{\partial f}{\partial x} = -s(s-y)(s-z) \qquad -s(s-y)(s-z) = \lambda \longrightarrow (s-z) = \frac{\lambda}{-s(s-y)}$

$\frac{\partial f}{\partial y} = -s(s-x)(s-z) \qquad (**)\ -s(s-x)(s-z) = \lambda \qquad -s(s-x)(s-z) = \frac{\lambda(s-x)}{(s-y)} = \lambda \ (*)$

$\frac{\partial f}{\partial z} = -s(s-x)(s-y) \qquad -s(s-x)(s-y) = \lambda \longrightarrow -s(s-x)^2 = \lambda$

ser at $\lambda \neq 0$. $(*) \Rightarrow \frac{s-x}{s-y} = 1 \Rightarrow x = y$.

$(s-x)^2 = -\frac{\lambda}{s}$

$(s-x) = \pm\sqrt{-\frac{\lambda}{s}}$.

$x = s \pm \sqrt{-\frac{\lambda}{s}}$

$s + \sqrt{-\frac{\lambda}{s}} > s$, så $x = s - \sqrt{-\frac{\lambda}{s}}$

sett inn i ligning $(**) \Rightarrow z = s - \sqrt{-\frac{\lambda}{s}}$

så $x = y = z = s - \sqrt{-\frac{\lambda}{s}}$.

5.10.16:

a) $P(x,y) = Kx^{\alpha}y^{\beta}$, $\underset{x+y=S}{\text{maks}}\ P(x,y)$

Lagrange gir $\frac{\partial P}{\partial x} = \lambda\frac{\partial g}{\partial x}$, $\frac{\partial P}{\partial y} = \lambda\frac{\partial g}{\partial y}$. $g(x,y) = x+y$. $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 1$.

Dette gir $\frac{\partial P}{\partial x} = \lambda = \frac{\partial P}{\partial y}$. Må altså finne et punkt hvor $\frac{\partial P}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y)$

$K\alpha x^{\alpha-1}y^{\beta} = K\beta x^{\alpha}y^{\beta-1}$

La $x = \frac{\alpha S}{\alpha+\beta}$, $y = \frac{\beta S}{\alpha+\beta}$. Da blir $\frac{\partial P}{\partial x}\left(\frac{\alpha S}{\alpha+\beta}, \frac{\beta S}{\alpha+\beta}\right) = K\alpha\left(\frac{\alpha S}{\alpha+\beta}\right)^{\alpha-1}\left(\frac{\beta S}{\alpha+\beta}\right)^{\beta}$

$= \frac{K\alpha^{\alpha}\beta^{\beta}S^{\alpha+\beta-1}}{(\alpha+\beta)^{\alpha+\beta-1}}$

$\frac{\partial P}{\partial y}\left(\frac{\alpha S}{\alpha+\beta}, \frac{\beta S}{\alpha+\beta}\right) = \frac{K\alpha^{\alpha}\beta^{\beta}S^{\alpha+\beta-1}}{(\alpha+\beta)^{\alpha+\beta-1}}$

b) La nå

$x_i = \frac{\alpha_i S}{\alpha_1+\dots+\alpha_n}$ $\qquad$ Sjekk hjemme!

12.1.3: $\boxed{\text{Hvis } |x|<1, \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$

a) $1-x+x^2-x^3+\dots = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$

siden

$|-x| = |x| < 1$

5. etasje NHA hus

b) $\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1-x^2}$

$|x|<1 \Rightarrow x^2 < 1$

$(|x^2|<1)$

c) $a^2 - 4a^4 + 16a^6 - \dots = \frac{1}{4}\left(4a^2 - (4a^2)^2 + (4a^2)^3 - \dots\right)$

$= -\frac{1}{4}\left(-4a^2 + (4a^2)^2 - (4a^2)^3 + \dots\right) = -\frac{1}{4}\sum_{n=1}^{\infty}(-4a^2)^n = -\frac{1}{4}\left(\sum_{n=0}^{\infty}(-4a^2)^n - 1\right)$

$\left(|a|<\frac{1}{2} \Rightarrow a^2 < \frac{1}{4} \Rightarrow |-4a^2| < 1\right)$

$= -\frac{1}{4}\left(\frac{1}{1-(-4a^2)} - 1\right) = -\frac{1}{4}\left(\frac{1}{1+4a^2} - \frac{1+4a^2}{1+4a^2}\right) = \frac{a^2}{1+4a^2}$

d) $\sum_{n=0}^{\infty} e^{-n/2} = \sum_{n=0}^{\infty}\left(e^{-\frac{1}{2}}\right)^n = \frac{1}{1-e^{-\frac{1}{2}}} = \frac{\sqrt{e}}{\sqrt{e}-1}$

$1 < e. \quad 1 < \sqrt{e} = e^{\frac{1}{2}}$

$0 < e^{-\frac{1}{2}} < 1.$