MAT 1110

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To første plenumsregninger → forelesninger
MATLAB

Gruppene starter neste uke:

Oblig < 23/2, 26/4 ← Underveiseksamen 30/3

Tøft kurs ⇒ jobb hardt.

Lett! vanskelig!

2.7 – 2.8, 1.9 – 1.10 ⇒ kap. 3 → kap. 6 → kap. 4 → kap. 5 ⇓ Kap 12 Kalkulus

Vanskelig! MATLAB

Repetisjon

$f : \mathbb{R}^n \to \mathbb{R}$ $\qquad y = f(x_1, \dots, x_n)$

$\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$ $\qquad \vec{y} = \vec{F}(x_1, \dots, x_n) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}$

Derivasjon: Partiell derivert.

$$\frac{\partial f}{\partial x_i} = \text{deriver mhp. } x_i \text{ som om de andre variablene er konstanter.}$$

Gradienten: $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$

Retningsderivert: $f'(\vec{a}, \vec{r}) = \nabla f(\vec{a}) \cdot \vec{r}$ forutsatt at f er deriverbar.

For vektorvaluerte funksjoner:

Jacobi-matrisen. $\vec{F}'(\vec{x}) = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \frac{\partial f_1}{\partial x_2}(\vec{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{x}) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}) & \frac{\partial f_2}{\partial x_2}(\vec{x}) & \dots & \frac{\partial f_2}{\partial x_n}(\vec{x}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}) & \frac{\partial f_m}{\partial x_2}(\vec{x}) & \dots & \frac{\partial f_m}{\partial x_n}(\vec{x}) \end{pmatrix}}_{m \times n\text{-matrise.}}$

$\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$

Deriverbarhet: $\vec{F}$ er deriverbar i $\vec{a}$ dersom

$$\vec{F}(\vec{a}+\vec{r}) = \vec{F}(\vec{a}) + \vec{F}'(\vec{a})\vec{r} + \vec{\sigma}(\vec{r}) \to 0 \text{ fortere enn } \vec{r}$$

dvs: $\vec{F}'(\vec{a})\vec{r}$ er en meget god tilnærmelse til $F(\vec{a}+\vec{r}) - \vec{F}(\vec{a})$.

Regneregler er som vanlig:

$$\left(\vec{F}(\vec{x}) + \vec{G}(\vec{x}) \right)' = \vec{F}(\vec{x}) + \vec{G}'(\vec{x})$$

$$\frac{\partial}{\partial x_i}\left(f(\vec{x}) g(\vec{x}) \right) = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}$$

Hva med kjerneregelen? Finnes det en kjerneregel for funksjoner av flere variable?

Ja! $\qquad$ Vanlig kjerneregel: $h(x) = f(g(x))$

$h'(x) = f'(g(x))\, g'(x)$

Kjerneregel i flere variable:

$$\vec{H}(\vec{x}) = \vec{F}(\vec{G}(\vec{x}))$$

$$\vec{H}'(x) = \vec{F}'(\vec{G}(\vec{x}))\, \vec{G}'(\vec{x})$$

Jacobi-matriser.

Kjerneregelen

Anta at vi har funksjonene: $f(u_1, u_2, \ldots, u_n)$

$g_1(x_1, x_2, \ldots, x_m)$

$g_2(x_1, x_2, \ldots, x_m)$

$\vdots$

$g_n(x_1, x_2, \ldots, x_m)$

Da kan vi danne en ny funksjon

$$h(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), g_2(x_1, \ldots, x_m), \ldots, g_n(x_1, x_2, \ldots, x_m))$$

Eksempel: $f(u_1, u_2) = 2u_1 u_2^2$ $\quad g_1(x_1, x_2, x_3) = x_1 + x_2 x_3^2$

$g_2(x_1, x_2, x_3) = x_2 + x_1 e^{x_3}$

$$h(x_1, x_2, x_3) = f(\underbrace{x_1 + x_2 x_3^2}_{u_1}, \underbrace{x_2 + x_1 e^{x_3}}_{u_2}) = 2(x_1 + x_2 x_3^2)(x_2 + x_1 e^{x_3})^2$$

Spørsmål: Kan vi regne ut de partiellderiverte til h dersom vi kjenner de partiellderiverte til $f, g_1, \ldots, g_n$?

$f(u_1, \ldots, u_n)$

$u_1 = g_1(x_1, \ldots, x_m)$

$\vdots$

Svar: Ja!

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial u_1}\frac{\partial g_1}{\partial x_i} + \frac{\partial f}{\partial u_2}\frac{\partial g_2}{\partial x_i} + \cdots + \frac{\partial f}{\partial u_n}\frac{\partial g_n}{\partial x_i}$$

Alle partiellderiverte av f

g'ene deriveres bare mhp. x_i

Eksempel: $f(u_1, u_2) = 2u_1 u_2^2$

$u_1 = g_1(x_1, x_2, x_3) = x_1 + x_2 x_3^2 \leftarrow h(x_1, x_2, x_3) = f(g_1(x_1, x_2, x_3), g_2(x_1, x_2, x_3))$

$u_2 = g_2(x_1, x_2, x_3) = x_2 + x_1 e^{x_3}$

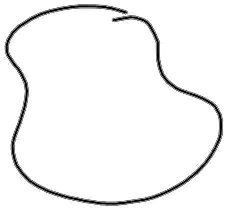
$$\frac{\partial h}{\partial x_2} = \frac{\partial f}{\partial u_1}\frac{\partial g_1}{\partial x_2} + \frac{\partial f}{\partial u_2}\frac{\partial g_2}{\partial x_2} = 2u_2^2 x_3^2 + 4u_1 u_2 \cdot 1$$

$$= 2(x_2 + x_1 e^{x_3})^2 x_3^2 + 4(x_1 + x_2 x_3^2)(x_2 + x_1 e^{x_3}).$$

Teorem (Kjerneregelen) Anta at funksjonene
$u_1 = g_1(x_1,\dots,x_m)$, $u_2 = g_2(x_1,\dots,x_m)$, $\dots$, $g_n(x_1,\dots,x_n)$
er deriverbare i punktet $\vec{x} = (x_1,\dots,x_n)$ og at funksjonen $f(u_1,\dots,u_n)$ er deriverbar i punktet $\vec{u} = (g_1(\vec{x}), g_2(\vec{x}),\dots,g_n(\vec{x}))$.
Da er den sammensatte funksjonen
$$h(x_1, x_2,\dots,x_m) = f(g_1(\vec{x}), g_2(\vec{x}),\dots, g_n(\vec{x}))$$
deriverbar i punktet $\vec{x}$ og
$$\frac{\partial h}{\partial x_i}(\vec{x}) = \frac{\partial f}{\partial u_1}(\vec{u})\frac{\partial g_1}{\partial x_i}(\vec{x}) + \frac{\partial f}{\partial u_2}(\vec{u})\frac{\partial g_2}{\partial x_i}(\vec{x}) + \dots + \frac{\partial f}{\partial u_n}(\vec{u})\frac{\partial g_n}{\partial x_i}(\vec{x})$$
$\vec{u} = (g_1(\vec{x}), g_2(\vec{x}),\dots,g_n(\vec{x}))$

Eksempel: Gasstank: $P = f(V,T)$

P = trykk
V = volum
T = temperaturen



Anta at vi vet hvordan ~~trykket~~ volumet og temperaturen endrer seg med tiden:

$V(t)$ — volum ved tid t

$T(t)$ — temperatur ved tid t.

Trykk ved tiden t: $P(t) = f(V(t), T(t))$

Endringen i trykket: $P'(t)$

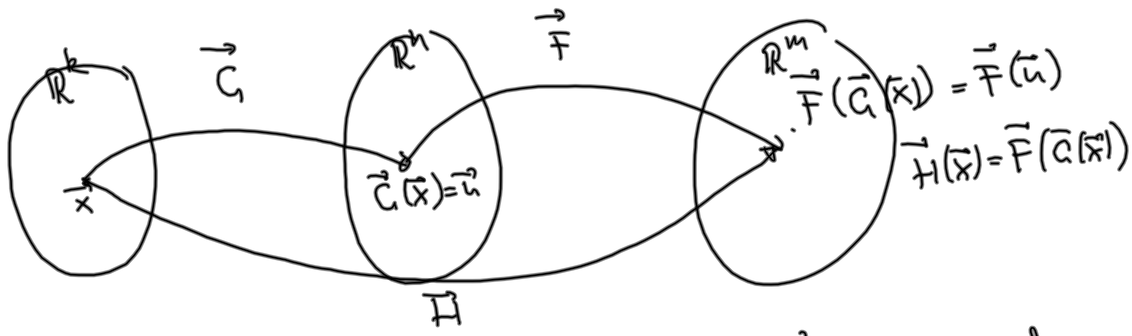
$$P'(t) = \frac{\partial P}{\partial t} = \frac{\partial f}{\partial V}\frac{\partial V}{\partial t} + \frac{\partial f}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial f}{\partial V}V'(t) + \frac{\partial f}{\partial T}T'(t)$$

To versjoner av kjerneregelen:

$$\vec{H}'(\vec{x}) = \vec{F}'(\vec{G}(x))\,\vec{G}'(x)$$

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial u_1}\frac{\partial g_1}{\partial x_i} + \frac{\partial f}{\partial u_2}\frac{\partial g_2}{\partial x_i} + \cdots + \frac{\partial f}{\partial u_n}\frac{\partial g_n}{\partial x_i}$$

Hva er sammenhengen?



$\vec{H}'(x)$ $m \times k$-matrise $\quad \vec{G}'(\vec{x})$ $n \times k$-matrise $\quad \vec{F}(\vec{u})$ $m \times n$-matrise

$$\underbrace{\vec{H}'(\vec{x})}_{m\times k} = \underbrace{\vec{F}(\vec{G}(\vec{x}))}_{m\times n}\ \underbrace{\vec{G}'(\vec{x})}_{n\times k}$$

$$\begin{array}{c c} & j \\ i & \begin{bmatrix} & \vdots & \\ \cdots & \frac{\partial H_i}{\partial x_j} & \cdots \\ & \vdots & \end{bmatrix} \end{array} = \begin{array}{c c} & \\ i & \begin{bmatrix} \frac{\partial F_i}{\partial u_1} & \frac{\partial F_i}{\partial u_2} & \cdots & \frac{\partial F_i}{\partial u_n} \\ & & & \end{bmatrix} \end{array} \begin{array}{c} j \\ \begin{bmatrix} \frac{\partial G_1}{\partial x_j} \\ \frac{\partial G_2}{\partial x_j} \\ \vdots \\ \frac{\partial G_n}{\partial x_j} \end{bmatrix} \end{array}$$

$$\boxed{\frac{\partial H_i}{\partial x_j} = \frac{\partial F_i}{\partial u_1}\frac{\partial G_1}{\partial x_j} + \frac{\partial F_i}{\partial u_2}\frac{\partial G_2}{\partial x_j} + \cdots + \frac{\partial F_i}{\partial u_n}\frac{\partial G_n}{\partial x_j}}$$