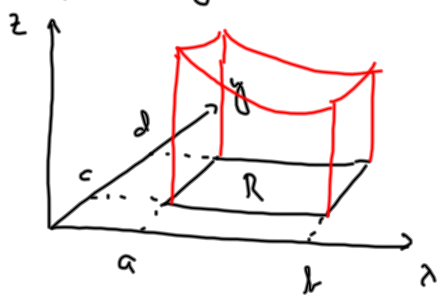
Dobbeltintegraler 6.2

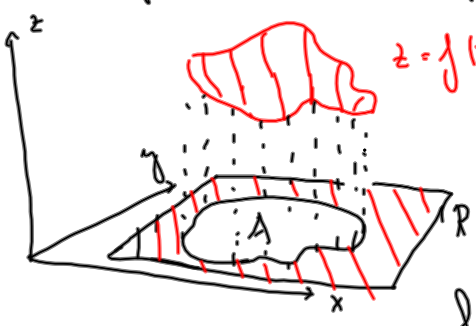
Forrige gang: Dobbeltintegraler over rektangulære områder;



$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right]dx$$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx\right]dy$$

Nå: Mer generelle integrasjonsområder. $z = f(x,y)$



Vi vil definere $\iint_A f(x,y)\,dx\,dy$

Vi innfører funksjonen f_A ved

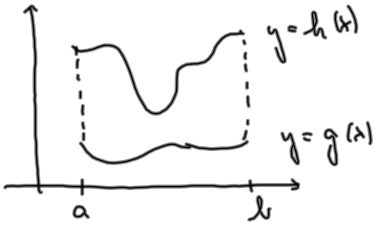
$$f_A(x,y) = \begin{cases} f(x,y) & \text{hvis } (x,y) \in A \\ 0 & \text{hvis } (x,y) \notin A. \end{cases}$$

Dersom R er et rektangel som inneholder A, så sier vi at f er integrerbar over A dersom f_A er integrerbar over R, og i så fall definerer vi

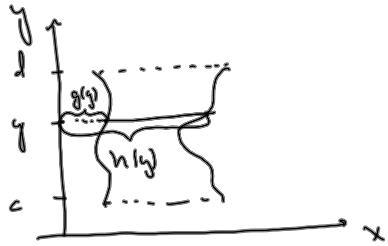
$$\iint_A f(x,y)\,dx\,dy = \iint_R f_A(x,y)\,dx\,dy.$$

I praksis regner man gjerne ut dobbeltintegraler over to typer områder:

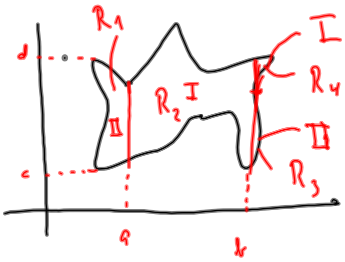
Type I:



Type II:

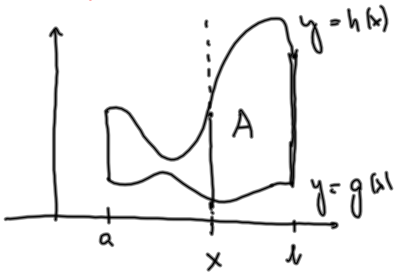


Kombinasjoner:

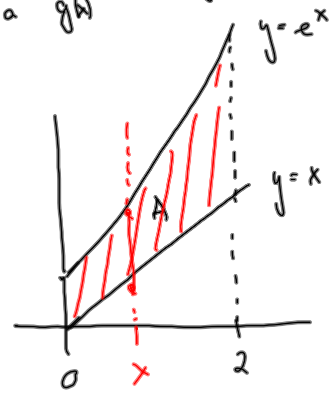


$$\iint_A = \iint_{R_1} + \iint_{R_2} + \iint_{R_3} + \iint_{R_4}$$

Integrasjon over områder av type I: Anta $h(x) \geq g(x)$ for alle $x \in [a,b]$



$$\int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy\right] dx$$



Eksempel: $f(x,y) = xy^2$

$$\iint_A f(x,y)\,dx\,dy = \int_0^2 \left[\int_x^{e^x} x y^2\,dy\right] dx$$

$$= \int_0^2 \left[x \frac{y^3}{3}\right]_{y=x}^{y=e^x} dx =$$

$$= \int_0^2 \left[x \frac{(e^x)^3}{3} - x\frac{x^3}{3}\right] dx = \frac{1}{3}\underbrace{\int_0^2 x e^{3x}\,dx}_{I_1} - \frac{1}{3}\underbrace{\int_0^2 x^4\,dx}_{I_2}$$

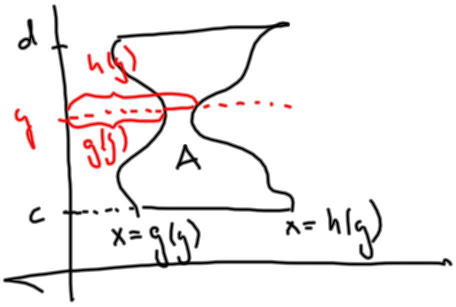
Mellomregning:

$$I_2 = \int_0^2 x^4\,dx = \left[\frac{x^5}{5}\right]_0^2 = \frac{32}{5}$$

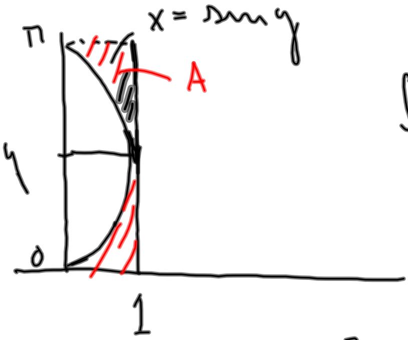
$$I_3 = \int_0^2 x e^{3x}\,dx = \left[\frac{1}{3} x e^{3x}\right]_0^2 - \int_0^2 \frac{1}{3} e^{3x}\,dx$$

$u = x$	$v' = e^{3x}$
$u' = 1$	$v = \frac{1}{3} e^{3x}$

Integrasjon av områder av type II



$$\iint_A f(x,y)\,dx\,dy = \int_c^d \left[\int_{g(y)}^{h(y)} f(x,y)\,dx \right] dy$$



$$f(x,y) = x$$

$$\iint_A x\,dx\,dy = \int_0^\pi \left[\int_{\sin y}^{1} x\,dx \right] dy$$

$$= \int_0^\pi \left[\frac{x^2}{2} \right]_{\sin y}^{1} dy = \int_0^\pi \left(\frac{1}{2} - \frac{\sin^2 y}{2} \right) dy$$

$$= \int_0^\pi \frac{1}{2}\,dy - \int_0^\pi \frac{\sin^2 y}{2}\,dy = \frac{\pi}{2} - \frac{1}{2} \underbrace{\int_0^\pi \sin^2 y\,dy}_{I} = \frac{\pi}{2} - \frac{\pi}{4} = \underline{\underline{\frac{\pi}{4}}}$$

<u>Mellomregning:</u> $\int_0^\pi \sin^2 y\,dy$

$$= \frac{1}{2}\int_0^\pi (1 - \cos 2y)\,dy = \frac{1}{2}\left[y - \frac{1}{2}\sin 2y \right]_0^\pi$$

$$= \frac{1}{2}\left[\pi - 0 - 0 - 0 \right] = \underline{\underline{\frac{\pi}{2}}}$$

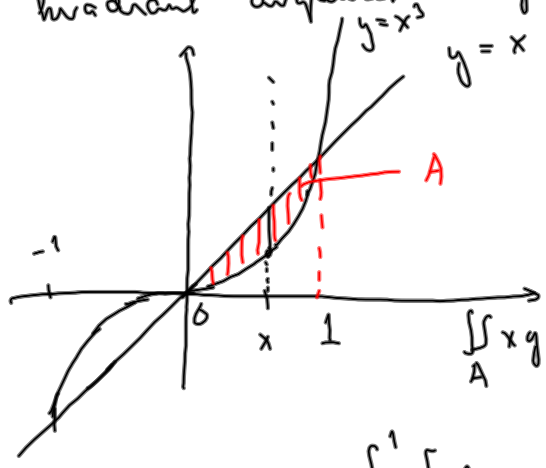
$$\cos 2x = \cancel{\cos^2 x - \sin^2 x}$$

$$= \cancel{2\cos^2 x - 1} = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

<u>Panikkhjelp:</u> Tirsdag 16 → ∞

Eksempel: Regn ut $\iint_A xy\,dx\,dy$ der A er området i første kvadrant avgrenset av grafene til $y=x$ og $y=x^3$.



Skjæring mellom grafene:

$$x^3 = x \iff x^3 - x = 0$$

$$\iff x(x^2-1) = 0 \iff x(x-1)(x+1) = 0$$

$$x=0,\ x=1,\ x=-1$$

$$\iint_A xy\,dx\,dy = \int_0^1 \left[\int_{x^3}^{x} xy\,dy\right] dx =$$

$$= \int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=x^3}^{y=x} dx = \int_0^1 \left[\tfrac{1}{2}x^3 - \tfrac{1}{2}x^7\right] dx$$

$$= \left[\frac{1}{8}x^4 - \frac{1}{16}x^8\right]_0^1 = \frac{1}{8} - \frac{1}{16} = \underline{\underline{\frac{1}{16}}}$$

Dobbeltintegraler i MATLAB: $R=[a,b]\times[c,d]$

dblquad(@(x,y) f(x,y), a, b, c, d)

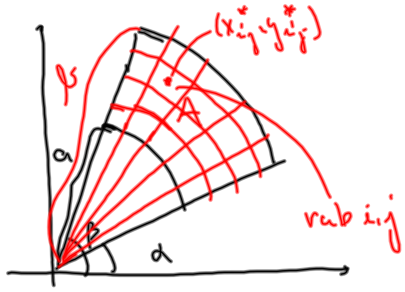
Indikatorfunksjon: $(f(x,y) <= g(x,y)) = \begin{cases} 1 & \text{hvis } f(x,y) \le g(x,y) \\ 0 & \text{ellers} \end{cases}$

Vi ønsker å integrere $f(x,y) = xy^2$ over ~~området~~

Vi ser på

$$xy^2 \cdot (x \le y) \cdot (y \le e^x)$$

Dobbeltintegraler i polarkoordinater

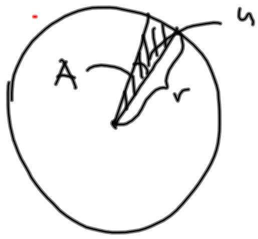


$$\iint_A f(x,y)\,dx\,dy \approx \sum_{i,j} f(x_{ij}^*, y_{ij}^*)\,|A_{ij}|$$

areal av biten A_{ij}

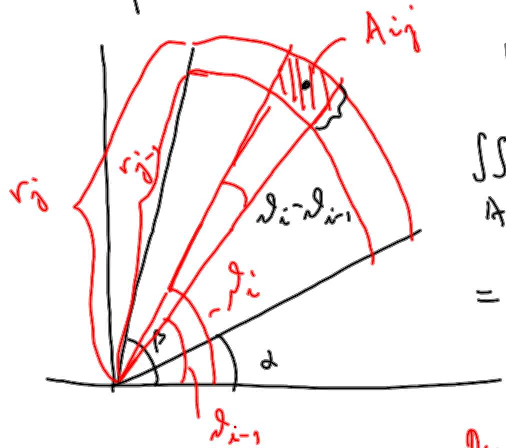
Hva er arealet til A_{ij}?

$$A = \frac{1}{2}R^2 u - \frac{1}{2}r^2 u$$
$$= \frac{1}{2}u(R^2 - r^2)$$
$$= \underbrace{\frac{1}{2}u(R+r)}(R-r) = r^* u(R-r)$$



$$\frac{A}{\pi r^2} = \frac{u}{2\pi} \Rightarrow A = \frac{\cancel{\pi} r^2 u}{2\cancel{\pi}} = \frac{1}{2}r^2 u$$

Vårt tilfelle



$$|A_{ij}| = r_{ij}^*(\theta_i - \theta_{i-1})(r_j - r_{j-1})$$

$$\iint_A f(x,y)\,dx\,dy \approx \sum f(x_{ij}^*, y_{ij}^*)\,r_{ij}^*(\theta_i - \theta_{i-1}) \cdot (r_j - r_{j-1})$$

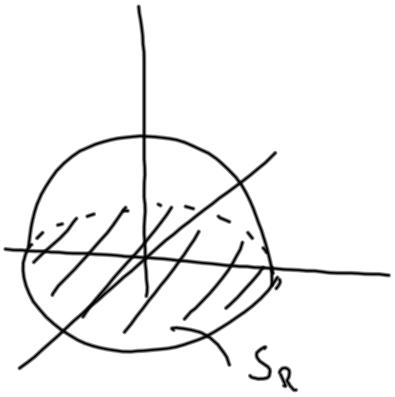
$$= \sum f(r_{ij}^*\cos\theta_{ij}^*, r_{ij}^*\sin\theta_{ij}^*)\, r_{ij}^*(\theta_i - \theta_{i-1})(r_j - r_{j-1})$$

Riemannsum til funksjonen $f(r\cos\theta, r\sin\theta)r$

$$\rightarrow \int_a^b\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr$$

<u>Eksempel</u>: Regn ut volumet til en halvkule med radius R.

$$x^2+y^2+z^2=R^2 \Rightarrow z=\sqrt{R^2-x^2-y^2}$$



$$V = \iint_{S_R} \sqrt{R^2-x^2-y^2}\,dx\,dy$$

$$= \int_0^R \int_0^{2\pi} \sqrt{R^2-r^2}\; r\,d\theta\,dr$$

$$u = R^2-r^2$$

$$du = -2r\,dr$$