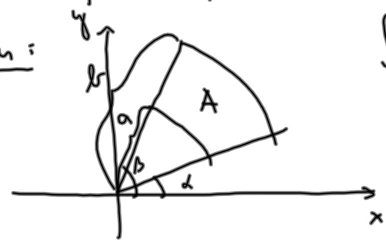
Integrasjon i polarkoordinater

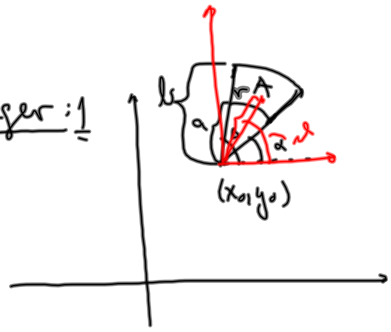
Grunnformelen:



$$\iint_A f(x,y)\,dx\,dy = \int_\alpha^\beta \Big[\int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\Big]\,d\theta$$

$$= \int_a^b \Big[\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\,d\theta\Big]\,dr$$

Generaliseringer: 1

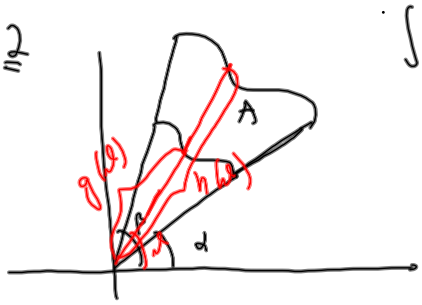


$x - x_0 = r\cos\theta \Rightarrow x = x_0 + r\cos\theta$

$y - y_0 = r\sin\theta \Rightarrow y = y_0 + r\sin\theta$

$$\iint_A f(x,y)\,dx\,dy = \int_\alpha^\beta \Big[\int_a^b f(x_0 + r\cos\theta,\ y_0 + r\sin\theta)\, r\,dr\Big]\,d\theta$$

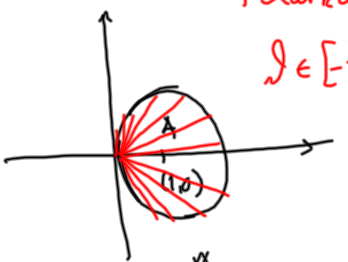
2



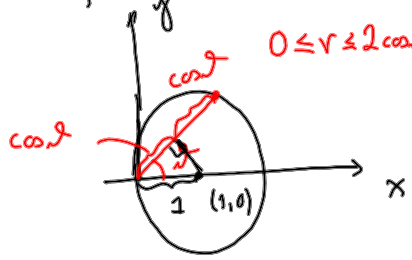
$$\iint_A f(x,y)\,dx\,dy = \int_\alpha^\beta \Big[\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\Big]\,d\theta$$

Polarkoordinater

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Eksempel: $\iint_A \sqrt{x^2+y^2}\,dx\,dy$



$0 \le r \le 2\cos\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big[\int_0^{2\cos\theta} \underbrace{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}}_{r}\; r\,dr\Big]\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big[\int_0^{2\cos\theta} r^2\,dr\Big]\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big[\frac{r^3}{3}\Big]_{r=0}^{r=2\cos\theta}\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}\cos^3\theta\,d\theta = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overbrace{\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}}^{1-u^2}\ \overbrace{\cos\theta\,d\theta}^{du}$$

$$= \frac{8}{3}\int_{-1}^{1} (1-u^2)\,du = \frac{8}{3}\Big[u - \frac{u^3}{3}\Big]_{-1}^{1} = \ldots\ldots$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$u\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

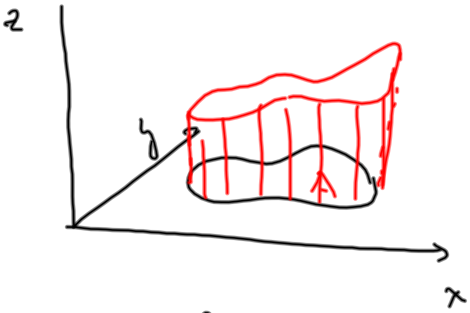
$u\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Anvendelser 6.4

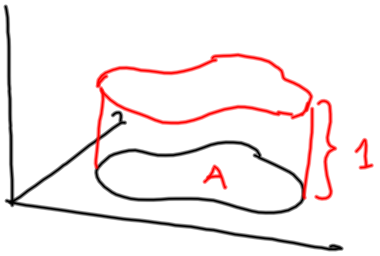
Volum: Hvis $f(x,y) \geq 0$ på A, så gir

$$V = \iint_A f(x,y)\,dx\,dy$$

volumet til legemet som ligger over A og under grafen til $z = f(x,y)$

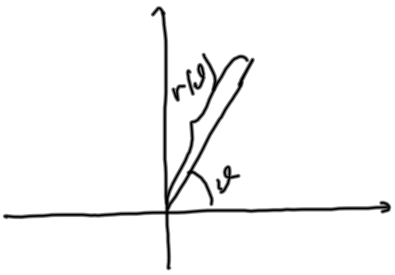


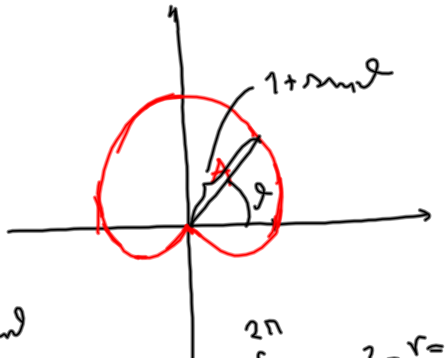
Areal:



$$\text{Areal} = \iint_A 1\,dx\,dy$$

Eksempel: Finn arealet omsluttet av kurven $r(\theta) = 1 + \sin\theta$, $0 \leq \theta \leq 2\pi$



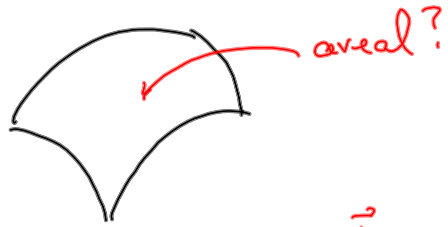


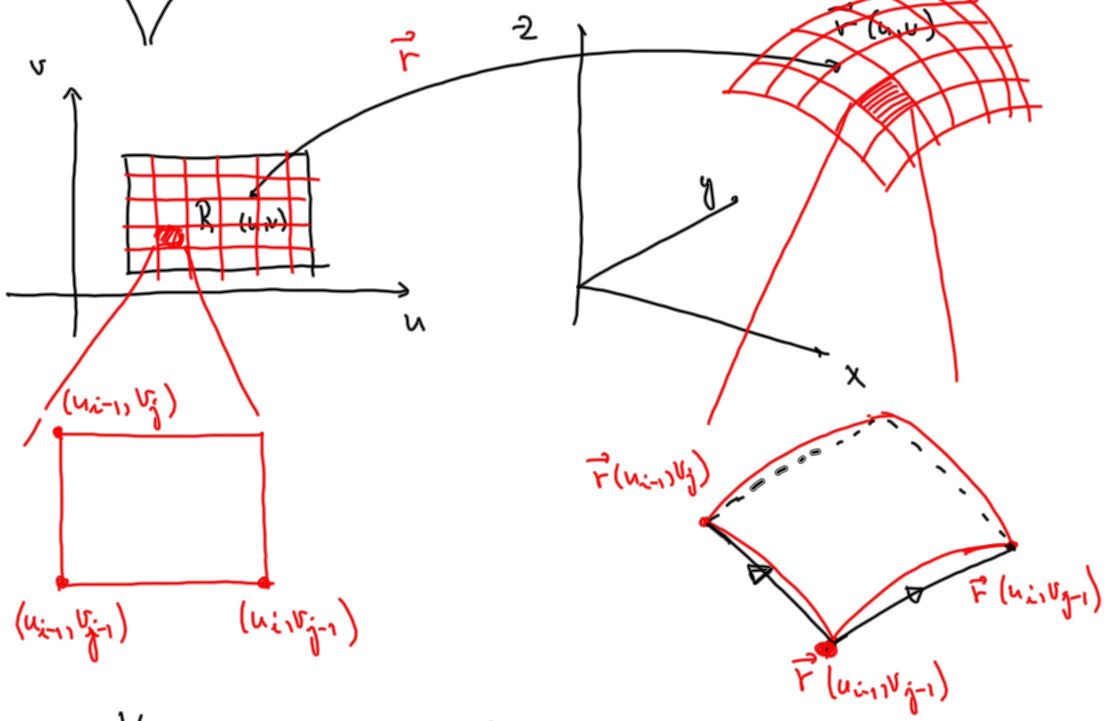
Areal: $$\iint_A 1\,dx\,dy = \int_0^{2\pi}\left[\int_0^{1+\sin\theta} \underbrace{1\cdot r}_{r}\,dr\right]d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=1+\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(1+\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 2\sin\theta + \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta - 2\cos\theta + \frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta\right]_0^{2\pi} = \ldots\ldots$$

Areal av flater





$$|A_{ij}| \approx \Big|\underbrace{\big(\vec{r}(u_i,v_{j-1}) - \vec{r}(u_{i-1},v_{j-1})\big)}_{\approx \frac{\partial\vec{r}}{\partial u}(u_{i-1},v_{j-1})(u_i-u_{i-1})} \times \underbrace{\big(\vec{r}(u_{i-1},v_j) - \vec{r}(u_{i-1},v_{j-1})\big)}_{\frac{\partial\vec{r}}{\partial v}(u_{i-1},v_{j-1})(v_j-v_{j-1})}\Big|$$

$$\approx \left|\frac{\partial\vec{r}}{\partial u}(u_{i-1},v_{j-1}) \times \frac{\partial\vec{r}}{\partial v}(u_{i-1},v_{j-1})\right|(u_i-u_{i-1})(v_j-v_{j-1})$$

Det totale arealet

$$A \approx \sum|A_{ij}| = \sum_{i,j}\underbrace{\left|\frac{\partial\vec{r}}{\partial u}(u_{i-1},v_{j-1}) \times \frac{\partial\vec{r}}{\partial v}(u_{i-1},v_{j-1})\right|(u_i-u_{i-1})(v_j-v_{j-1})}_{\text{Riemannsum til } \left|\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)(u,v)\right|}$$

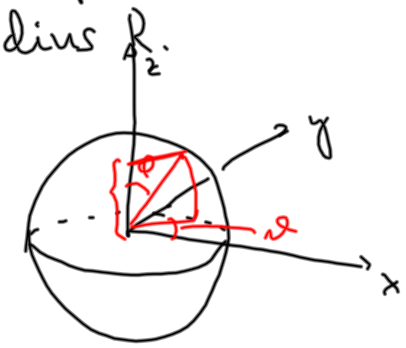
$$\longrightarrow \iint_R \left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|(u,v)\,du\,dv$$

<u>Definisjon</u>: Arealet til flaten parametrisert ved $\vec{r}: A\to\mathbb{R}^3$ er gitt ved

$$\iint_A \left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|(u,v)\,du\,dv$$

forutsatt at integralet eksisterer.

Eksempel: Arealet til overflaten til en kule med radius R.



$$\theta \in [0, 2\pi)$$
$$\varphi \in [0, \pi]$$

$$\vec{r}(\theta, \varphi) = R\sin\varphi\cos\theta\,\vec{i} + R\sin\varphi\sin\theta\,\vec{j} + R\cos\varphi\,\vec{k}$$

$$\frac{\partial\vec{r}}{\partial\varphi} = R\cos\varphi\cos\theta\,\vec{i} + R\cos\varphi\sin\theta\,\vec{j} - R\sin\varphi\,\vec{k}$$

$$\frac{\partial\vec{r}}{\partial\theta} = -R\sin\varphi\sin\theta\,\vec{i} + R\sin\varphi\cos\theta\,\vec{j} + 0\vec{k}$$

$$\frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi \\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= R^2\sin^2\varphi\cos\theta\,\vec{i} + R^2\sin^2\varphi\sin\theta\,\vec{j} + \left(\underline{R^2\cos\varphi\sin\varphi}\cos^2\theta + \underline{R^2\cos\varphi\sin\varphi}\sin^2\theta\right)\vec{k}$$

$$= R^2\sin^2\varphi\cos\theta\,\vec{i} + R^2\sin^2\varphi\sin\theta\,\vec{j} + R^2\cos\varphi\sin\varphi\,\vec{k}$$

$$\left|\frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial\theta}\right| = \sqrt{R^4\underline{\sin^4\varphi}\cos^2\theta + R^4\underline{\sin^4\varphi}\sin^2\theta + R^4\cos^2\varphi\sin^2\varphi}$$

$$= R^2\sqrt{\underbrace{\sin^4\varphi}_{\sin^2\varphi\sin^2\varphi} + \cos^2\varphi\sin^2\varphi} = R^2\sqrt{\sin^2\varphi} = R^2\sin\varphi$$

Arealet:

$$\int_0^{\pi}\int_0^{2\pi}\left|\frac{\partial r}{\partial\varphi} \times \frac{\partial r}{\partial\theta}\right| d\theta\, d\varphi = \int_0^{\pi}\left[\int_0^{2\pi} R^2\sin\varphi\, d\theta\right] d\varphi$$

$$= \int_0^{\pi} 2\pi R^2\sin\varphi\, d\varphi = 2\pi R^2\left[-\cos\varphi\right]_0^{\pi} = 2\pi R^2\overbrace{\left[-(-1)-(-1)\right]}^{2}$$

$$= \underline{\underline{4\pi R^2}}$$