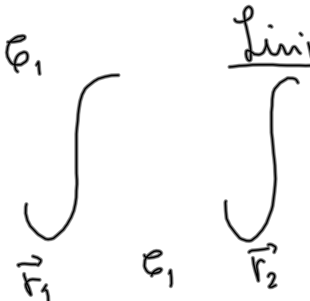
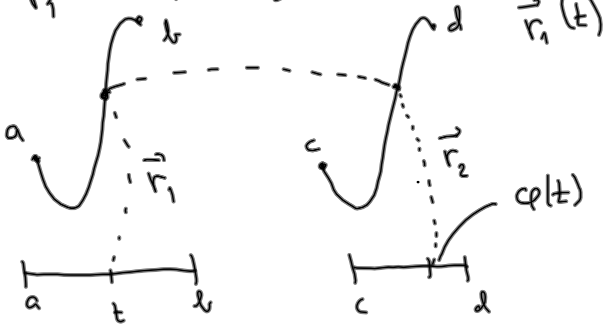
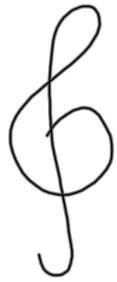
Linjeintegraler



$$\int_{\mathcal{C}_1} f\,ds \overset{?}{=} \int_{\mathcal{C}_2} f\,ds$$



$\vec{r}_1(t) = \vec{r}_2(\varphi(t))$



Hvordan må φ være:

(i) $\varphi(a) = c \quad \varphi(b) = d$

(ii) φ er en strengt voksende funksjon

(iii) φ er en deriverbar funksjon der $\varphi'(x) > 0$ for alle x

(iv) $\vec{r}_1(t) = \vec{r}_2(\varphi(t))$

I så fall kalles parametriseringen ekvivalente.

Setning: Bruker vi ekvivalente parametriseringer til å regne ut et linjeintegral, får vi samme svar, dvs $\int_{\mathcal{C}_1} f\,ds = \int_{\mathcal{C}_2} f\,ds$.

Bevis: Vet at $\vec{r}_1(t) = \vec{r}_2(\varphi(t))$

$$\vec{r}_1{}'(t) = \vec{r}_2{}'(\varphi(t))\,\varphi'(t)$$

$$v_1(t) = |\vec{r}_1{}'(t)| = |\vec{r}_2{}'(\varphi(t))\,\varphi'(t)| = |\vec{r}_2{}'(\varphi(t))|\,\varphi'(t)$$
$$= v_2(\varphi(t))\,\varphi'(t)$$

Vi har

$$\int_{\mathcal{C}_1} f\,ds = \int_a^b f(\vec{r}_1(t))\,v_1(t)\,dt = \int_a^b f(\vec{r}_2(\underbrace{\varphi(t)}_{u}))\,v_2(\underbrace{\varphi(t)}_{u})\,\overbrace{\varphi'(t)\,dt}^{du}$$

$$= \int_c^d f(\vec{r}_2(u))\,v_2(u)\,du = \int_{\mathcal{C}_2} f\,ds$$

$u = \varphi(t)$

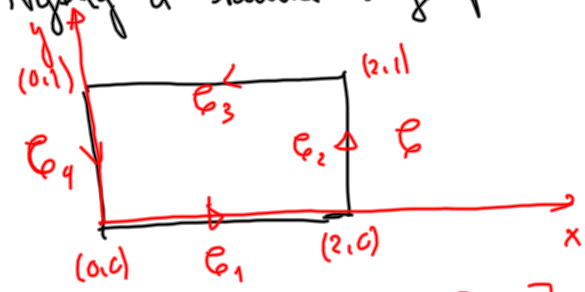
$du = \varphi'(t)\,dt$

$c = \varphi(a)$

$d = \varphi(b)$

Det viser seg at også motsatt rettede parametriseringer gir samme svar for $\int_{\mathcal{C}} f\,ds$

Nyttig å kunne velge parametrisering selv.



$$\int_{\mathcal{C}} f\,ds = \int_{\mathcal{C}_1} f\,ds + \int_{\mathcal{C}_2} f\,ds + \int_{\mathcal{C}_3} f\,ds + \int_{\mathcal{C}_4} f\,ds$$

$$\vec{r}_1(t) = t\,\vec{i} = (t,0), \quad t \in [0,2]$$

$$\vec{r}_2(t) = 2\,\vec{i} + t\,\vec{j} = (2,t) \qquad t \in [0,1]$$

$$\vec{r}_3(t) = t\,\vec{i} + 1\,\vec{j} = (t,1) \qquad t \in [0,2]$$

$$r_4(t) = t\,\vec{j} = (0,t) \qquad t \in [0,1]$$

Eksempel: Regn ut $\int_{\mathcal{C}} xy^2\,ds$ når $\mathcal{C}$ er som ovenfor.

$$\int_{\mathcal{C}_1} f\,ds = \int_0^2 f(\vec{r}_1(t))\,v_1(t)\,dt = \int_0^2 t\cdot 0^2\,v_1(t)\,dt = 0$$

$$\vec{v}_2(t) = (0,1)$$

$$v_2(t) = |\vec{v}_2(t)| = 1$$

$$\int_{\mathcal{C}_2} f\,ds = \int_0^1 f(\vec{r}_2(t))\,v_2(t)\,dt = \int_0^1 2t^2\,v_2(t)\,dt$$

$$= \int_0^1 2t^2\,dt = \left[\tfrac{2}{3}t^3\right]_0^1 = \underline{\underline{\tfrac{2}{3}}}$$

$$\int_{\mathcal{C}_3} f\,ds = \int_0^2 f(\vec{r}_3(t))\,v_3(t)\,dt = \int_0^2 t\cdot 1^2\cdot 1\,dt \qquad \vec{v}_3(t) = (1,0)$$

$$v_3(t) = 1$$

$$= \left[\tfrac{t^2}{2}\right]_0^2 = \underline{\underline{2}}$$

$$\int_{\mathcal{C}_4} f\,ds = \int_0^1 f(\vec{r}_4(t))\,v_4(t) = \int_0^1 0\cdot t^2\,v_4(t)\,dt = 0$$

Tilsl: $\int_{\mathcal{C}} f\,ds = 0 + \frac{2}{3} + 2 + 0 = \underline{\underline{\frac{8}{3}}}$.