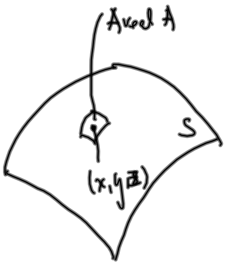
Flateintegraler (6.4)

Linjeintegraler: $\int_{\mathcal{C}} f\, ds\,,\quad \int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$

Flateintegraler: $\int_S f\, dS\,,\quad \int_S \vec{F}\cdot\vec{n}\, dS$

flateintegral av skalarfelt.

Intuisjon:



masse $\approx f(x,y,z)\cdot A$

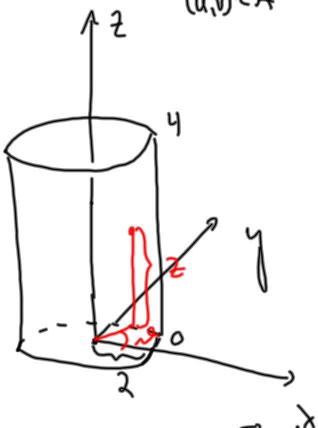
Total masse: $\int_S f\, dS$

Definisjon: $\int_S f\, dS$

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

$(u,v)\in A$

$$= \iint_A f(\vec{r}(u,v))\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|(u,v)\, du\, dv$$



Eksempel: $\int_S x^2yz\, dS$ $\quad S$

$\vec{r}(\theta,z) = 2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j} + z\vec{k}\,,\quad \theta\in[0,2\pi]$

$z\in[0,4]$

$\frac{\partial\vec{r}}{\partial\theta} = -2\sin\theta\,\vec{i} + 2\cos\theta\,\vec{j} + 0\vec{k}$

$\frac{\partial\vec{r}}{\partial z} = 0\vec{i} + 0\vec{j} + 1\vec{k}$

$$\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial z} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j} + 0\vec{k}$$

$$\left|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial z}\right| = \sqrt{4\cos^2\theta + 4\sin^2\theta + 0^2} = \underline{\underline{2}}$$

$$\int_S x^2yz\, dS = \iint_A (2\cos\theta)^2(2\sin\theta)z\,\overbrace{\left|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial z}\right|}^{2}\, d\theta\, dz$$

$$= 16\int_0^4\left[\int_0^{2\pi}\cos^2\theta\sin\theta\, z\, d\theta\right]dz = 16\int_0^4\left[-\frac{\cos^3\theta}{3}z\right]_{\theta=0}^{2\pi}dz = \underline{\underline{0}}$$

Greens teorem (16.5)

Linjeintegraler i $\mathbb{R}^2$: $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \ t \in [a,b]$

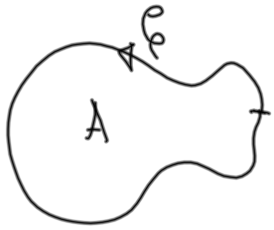
$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_a^b \Big(\underbrace{P(x(t),y(t))\vec{i} + Q(x(t),y(t))\vec{j}}_{\vec{F}(\vec{r}(t))}\Big)\cdot \underbrace{\big(x'(t)\vec{i} + y'(t)\vec{j}\big)}_{\vec{r}\,'(t)}\, dt$$

$$= \int_a^b P(x(t),y(t))\cdot \underbrace{x'(t)\,dt}_{dx} + \int_a^b Q(x(t),y(t))\,\underbrace{y'(t)\,dt}_{dy}$$

$$= \int_{\mathcal{C}} P\,dx + Q\,dy$$ (alternativt uttrykk for $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$)

Enkel lukket kurve:

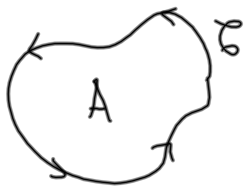


starter og ender i samme punkt, skjærer ikke seg selv underveis.

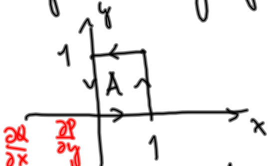
Greens teorem: Anta at $\mathcal{C}$ er en enkel lukket kurve som avgrenser et område A, og som har en stykkevis glatt parametrisering $\vec{r}(t)$. Dersom $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ er et vektorfelt som har kontinuerlige partiellderiverte i et område som inneholder A, så er

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

når $\mathcal{C}$ har positiv orientering.



Eksempel: Regn ut $\int_{\mathcal{C}} x^3 y\,dx + (x+y)\,dy$ der $\mathcal{C}$ er kvadratet



$P(x,y) = x^3 y$; $\frac{\partial P}{\partial y} = x^3$

$Q(x,y) = x+y$; $\frac{\partial Q}{\partial x} = 1$

$$\int_{\mathcal{C}} \overbrace{x^3 y}^{P}\,dx + \overbrace{(x+y)}^{Q}\,dy = \iint_A (\underbrace{1}_{\frac{\partial Q}{\partial x}} - \underbrace{x^3}_{\frac{\partial P}{\partial y}})\,dx\,dy = \int_0^1\left[\int_0^1 (1-x^3)\,dy\right]dx$$

$$= \int_0^1 (1-x^3)\,dx = \left[x - \frac{x^4}{4}\right]_0^1 = \underline{\underline{\frac{3}{4}}}$$

Arealberegning ved hjelp av Greens teorem:

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\mathcal{C}} P\,dx + \int_{\mathcal{C}} Q\,dy$$

Velg $Q = x$, $P = 0$:

$$\text{areal}(A) = \iint_A \underbrace{(1-0)}_{1}\,dx\,dy = \int_{\mathcal{C}} x\,dy$$

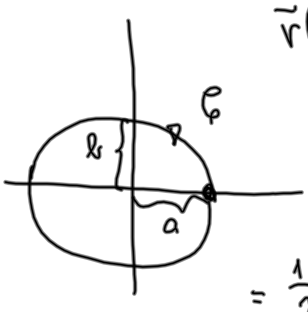
Velg $Q = 0$, $P = -y$

$$\text{areal}(A) = \iint_A (0+1)\,dx\,dy = -\int_{\mathcal{C}} y\,dx$$

$$\text{areal}(A) = \frac{1}{2}\text{areal}(A) + \frac{1}{2}\text{areal}(A) = \frac{1}{2}\int_{\mathcal{C}} x\,dy - \frac{1}{2}\int_{\mathcal{C}} y\,dx = \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx$$

Eksempel: Arealet av ellipsen: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\vec{r}(t) = a\cos t\,\vec{i} + b\sin t\,\vec{j} \qquad t \in [0, 2\pi]$$



$$\text{areal}(A) = \frac{1}{2}\int -y\,dx + x\,dy$$

$x(t) = a\cos t$, $dx = -a\sin t\,dt$
$y(t) = b\sin t$, $dy = b\cos t\,dt$

$$= \frac{1}{2}\int_0^{2\pi} -\underbrace{b\sin t}_{y}\,\underbrace{(-a\sin t)\,dt}_{dx} + \underbrace{a\cos t}_{x}\,\underbrace{b\cos t\,dt}_{dy}$$

$$= \frac{1}{2}\int_0^{2\pi} \underbrace{(ab\sin^2 t + ab\cos^2 t)}_{ab}\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{\cancel{2}}\,\cancel{2}\pi ab = \underline{\underline{\pi ab}}$$

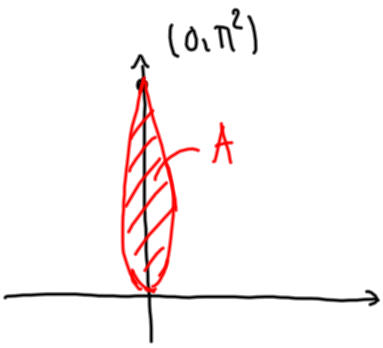
"Baklengs bruk" av Greens teorem: fra dobbeltintegral til linjeintegral.

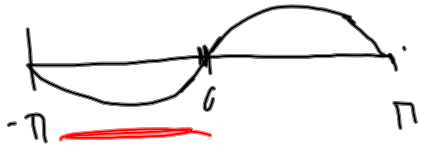
$$\iint_A \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\iint f(x,y)\,dx\,dy} dx\,dy = \int_C P\,dx + Q\,dy$$

$\iint f(x,y)\,dx\,dy$ må velge P, Q slik at dette stemmer.

Eksempel: $\iint_A y\,dx\,dy$ der A er området avgrenset av kurven

$\vec{r}(t) = \sin t\,\vec{i} + t^2\,\vec{j}$, $t \in [-\pi, \pi]$: C





Idé: $\iint_A \underbrace{y}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}\,dx\,dy = \int_C P\,dx + Q\,dy$

$P = 0$: $\frac{\partial Q}{\partial x} = y \Rightarrow Q = xy$

$$\iint_A y\,dx\,dy = \iint_A \Big(\underbrace{\frac{\partial Q}{\partial x}}_{y} - \underbrace{\frac{\partial P}{\partial y}}_{0}\Big)dx\,dy = \int_C 0\,dx + xy\,dy = \int_C xy\,dy =$$

$$= \int_{-\pi}^{\pi} \underbrace{\sin t}_{x} \cdot \underbrace{t^2}_{y} \cdot 2t\,dt = 2\int_{-\pi}^{\pi} t^3 \sin t\,dt$$

Husk: $\vec{r}(t) = \sin t\,\vec{i} + t^2\,\vec{j}$, $t \in [-\pi, \pi]$

$dy = 2t\,dt$

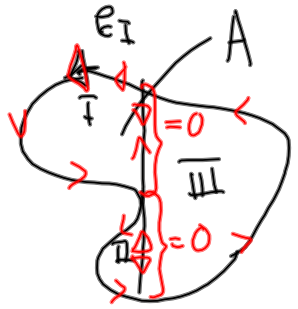
$u = t^3$ $v' = \sin t$

$u' = 3t^2$ $v = -\cos t$

osv.

$= \ldots\ldots\ldots$

Oppdeling av områder: Anta at Greens teorem gjelder for hvert av områdene I, II og III



$$\iint_{I} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{C_I} P\,dx + Q\,dy$$

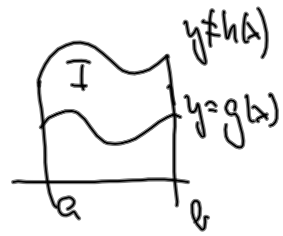
$$\iint_{II} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{C_{II}} \cdots$$

$$\iint_{III} \text{—} \, '' \, \text{—} = \int_{C_{III}}$$

Legger sammen $$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= \int_C P\,dx + Q\,dy$$

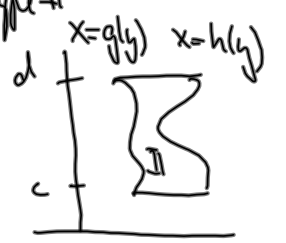
Greens teorem består av en P og en Q-del.



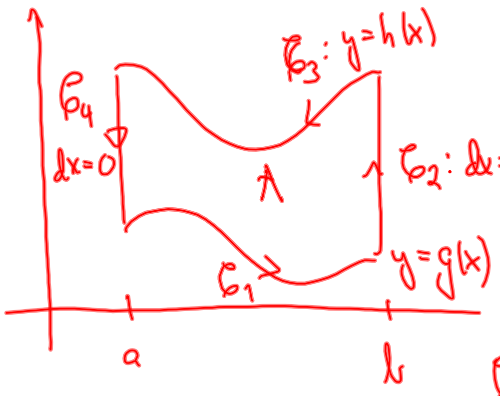
$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_C P\,dx + Q\,dy$$

P-del: $\iint_A -\frac{\partial P}{\partial y}\,dx\,dy = \int_C P\,dx$ = lett å vise for områder av type I

Q-del: $\iint_A \frac{\partial Q}{\partial x}\,dx\,dy = \int_C Q\,dy$ = —— '' —— type II



Skal vise P-delen for område I.



$$\iint_A -\frac{\partial P}{\partial y}\,dx\,dy = \int_{\mathcal{C}} P\,\underline{dx}$$

$$= \int_{\mathcal{C}_1} P\,dx + \int_{\mathcal{C}_3} P\,dx$$

$\mathcal{C}_1: \vec{r}_1(t) = t\,\vec{i} + g(t)\,\vec{j} \qquad t \in [a,b]$

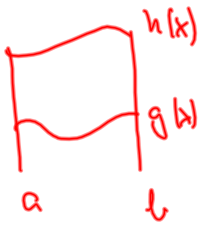
$d\vec{r}_1(t) = \vec{i} + g'(t)\,\vec{j}, \quad \underbrace{dx = dt}$

$\mathcal{C}_3: \vec{r}_3(t) = t\,\vec{i} + h(t)\,\vec{j}, \quad t \in [a,b]$ OBS: Wrong way!

$d\vec{r}_3(t) = \vec{i} + h'(t)\,\vec{j} \quad \underset{\smile}{dx} = dt$

$$\int_{\mathcal{C}_1} P\,dx = \int_a^b P(t, g(t))\,dt \,; \quad \int_{\mathcal{C}_3} P\,dx = -\int_a^b P(t, h(t))\,dt$$

$$\int_{\mathcal{C}} P\,dx = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_3} = \int_a^b P(t, g(t))\,dt - \int_a^b P(t, h(t))\,dt$$



<u>Sammenlign med:</u> $\displaystyle \iint_A -\frac{\partial P}{\partial y}\,dx\,dy = -\int_a^b \left[\int_{g(x)}^{h(x)} \frac{\partial P}{\partial y}\,dy\right] dx$

$$= -\int_a^b \left[P(x,y)\right]_{y=g(x)}^{y=h(x)} dx = -\int_a^b \left[P(x,h(x)) - P(x,g(x))\right] dx$$

$$= \int_a^b \left[P(x,g(x)) - P(x,h(x))\right] dx$$

<u>Hurra</u>: $\displaystyle \iint_A -\frac{\partial P}{\partial y}\,dx\,dy = \int_{\mathcal{C}} P\,dx$