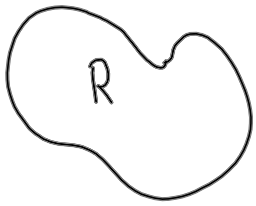
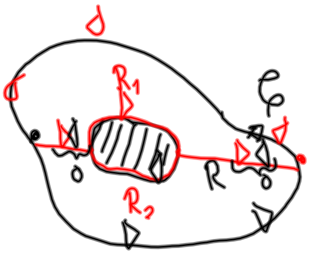
Greens teorem



$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\mathcal{C}} P\,dx + Q\,dy$$

Hva skjer hvis det er et hull i R? Vanlig Green på R_1 og R_2:



$$\iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\mathcal{C}_1} P\,dx + Q\,dy$$

$$\iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\mathcal{C}_2} P\,dx + Q\,dy$$

P ikke f

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{R_1} + \iint_{R_2} = \int_{\mathcal{C}_1} P\,dx + Q\,dy + \int_{\mathcal{C}_2} P\,dx + Q\,dy$$

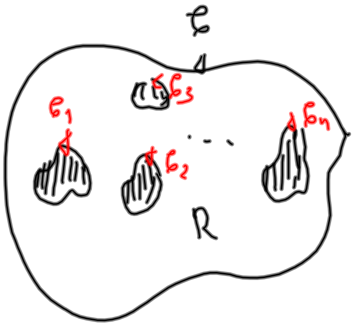
$$= \int_{\mathcal{C}} P\,dx + Q\,dy - \int_{D} P\,dx + Q\,dy$$

ytre rand til R

randen til hullet med positiv orientering

Mer generelt:

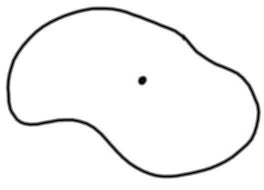
$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\mathcal{C}} P\,dx + Q\,dy - \int_{\mathcal{C}_1} P\,dx + Q\,dy - \int_{\mathcal{C}_2} P\,dx + Q\,dy - \cdots - \int_{\mathcal{C}_n} P\,dx + Q\,dy$$



ADVARSEL: Hvis $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ ikke er definert i ett eneste punkt i R, risikerer vi at

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\mathcal{C}} P\,dx + Q\,dy$$

ikke holder.



Skifte av variabel (6.7)

En variabel: $u = g(x) \Rightarrow x = h(u)$, $dx = h'(u)\,du$

$$\int_a^b f(x)\,dx = \int_{g(a)}^{g(b)} f(\underbrace{h(u)}_{\text{ny variabel}})\underbrace{h'(u)}_{\text{justeringsfaktor}}\,du$$

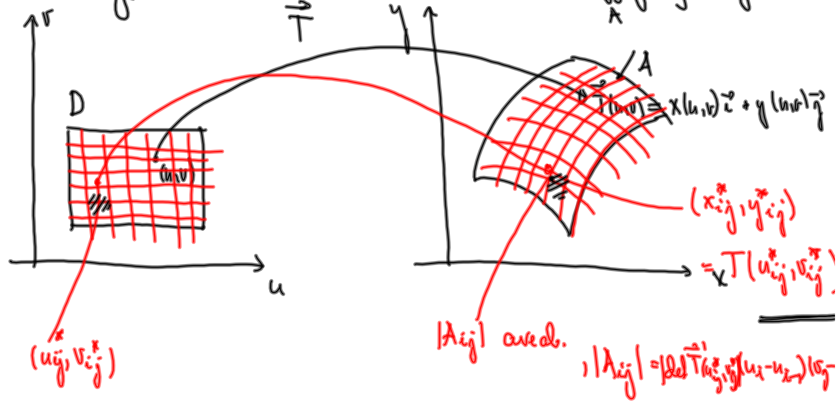
nytt integrasjons-område.

To variable:

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(\underbrace{x(u,v),y(u,v)}_{\text{nye variable}})\underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{justeringsfaktor}}\,du\,dv$$

nytt område

Forklaring:



$$\iint_A f(x,y)\,dx\,dy \approx \sum_{i,j} f(x_{ij}^*, y_{ij}^*)|A_{ij}| = \sum_{i,j} f(\vec{T}(u_{ij}^*, v_{ij}^*))|A_{ij}|$$

$$= \sum_{i,j} \underbrace{f(\vec{T}(u_{ij}^*, v_{ij}^*))\,|\det \vec{T}'(u_{ij}^*, v_{ij}^*)|\,(u_i - u_{i-1})(v_j - v_{j-1})}_{\text{Riemann-sum } f(\vec{T}(u,v))\,|\det \vec{T}'(u,v)|}$$

$$\longrightarrow \iint_D f(\vec{T}(u,v))\,|\det \vec{T}'(u,v)|\,du\,dv$$

Formel for skifte av variabel:

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(\vec{T}(u,v))\,|\det \vec{T}'(u,v)|\,du\,dv$$

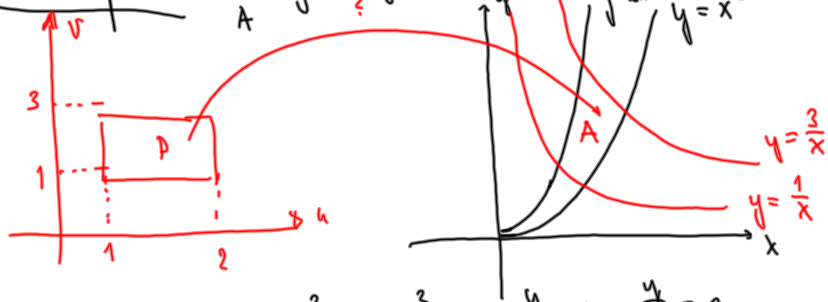
Avbildning: $\vec{T}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}$

$$\vec{T}'(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

notasjon!

$\det \vec{T}'(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \partial(x,y)$

Eksempel: $\iint\limits_A x^2y\,dx\,d\underset{?}{y}$ der A er området



Avgrensninger: $y=x^2,\ y=2x^2 \Rightarrow \frac{y}{x^2}=1,\ \frac{y}{x^2}=2$

$1\le \frac{y}{x^2}\le 2 \qquad u=\frac{y}{x^2}$

$y=\frac{1}{x},\ y=\frac{3}{x} \Rightarrow xy=1,\ xy=3$

$1\le xy\le 3 \qquad v=xy$

Velger nye variable: $u=\frac{y}{x^2},\ v=xy$ og får da området $1\le u\le 2$, $1\le v\le 3$

Må finne x og y uttrykt ved u og v.

$u=\frac{y}{x^2},\ v=xy$

$\Downarrow$

$y=ux^2 \qquad v=xux^2 \Rightarrow x^3=\frac{v}{u} \Rightarrow x=\sqrt[3]{\frac{v}{u}}=u^{-1/3}v^{1/3}$

$y=ux^2=u\,u^{-2/3}v^{2/3}=u^{1/3}v^{2/3}$

Altså $x=u^{-1/3}v^{1/3},\ y=u^{1/3}v^{2/3}$

Jacobi-determinanten: $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$

$$=\begin{vmatrix}-\frac{1}{3}u^{-\frac{4}{3}}v^{1/3} & \frac{1}{3}u^{-1/3}v^{-2/3}\\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3}\end{vmatrix}=-\frac{2}{9}u^{-1}-\frac{1}{9}u^{-1}=-\frac{3}{9}u^{-1}=\underline{-\frac{1}{3}u^{-1}}$$

Variabelskifte: $\iint\limits_A f(x,y)\,dx\,dy=\iint\limits_D f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$

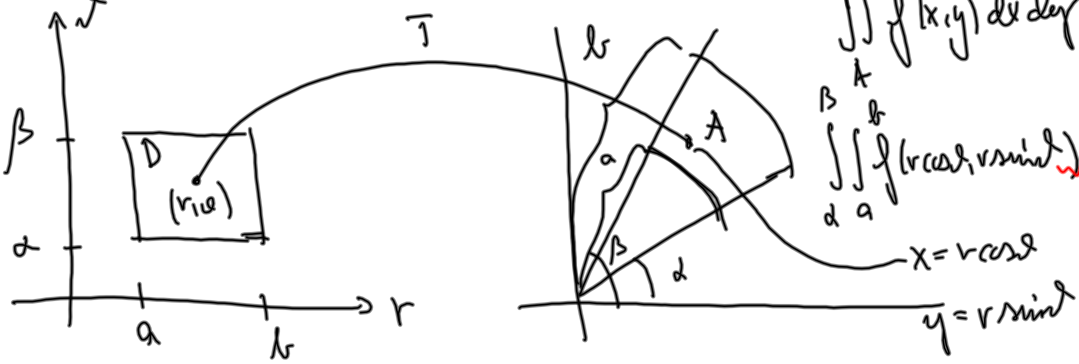
$$\iint\limits_A x^2y\,dx\,dy=\iint\limits_D \left(u^{-1/3}v^{1/3}\right)^2u^{1/3}v^{2/3}\left|-\frac{1}{3}u^{-1}\right|du\,dv$$

$u^{-\frac{2}{3}}u^{1/3}u^{-1}=u^{-}$

$v^{2/3}v^{2/3}=v^{4/3}$

$$=\int_1^3\left[\int_1^2\frac{1}{3}u^{-4/3}v^{4/3}\,du\right]dv=\int_1^3\frac{1}{3}\left[3u^{-1/3}\right]_{u=1}^{u=2}v^{4/3}\,dv \ldots$$

Polarkoordinater



$$\iint_A f(x,y)\,dx\,dy =$$

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Shifte av variabel:

$$\iint_A f(x,y)\,dx\,dy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \underbrace{\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}_{r} dr\,d\theta$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

der

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r$$