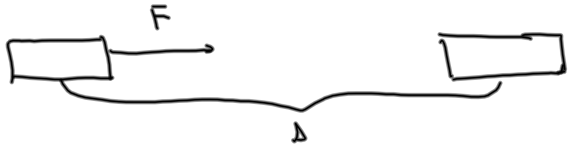
Linjeintegraler av vektorfelt

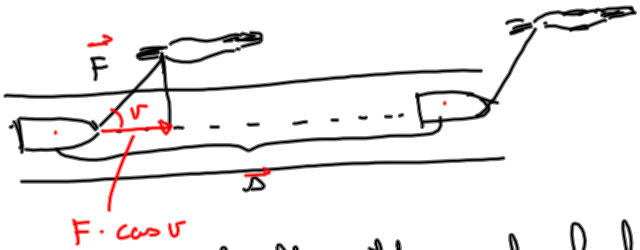
Sist: Linjeintegraler av skalarfelt: $\int_{\mathcal{C}} f\,ds$, $f:\mathbb{R}^n\to\mathbb{R}$

Nå: —— " —— vektorfelt: $\int_{\mathcal{C}} \vec F\cdot d\vec r$, $\vec F:\mathbb{R}^n\to\mathbb{R}^n$

Lynkurs: Ungdomsskolen: Arbeid = kraft × vei



$A = F\cdot s$

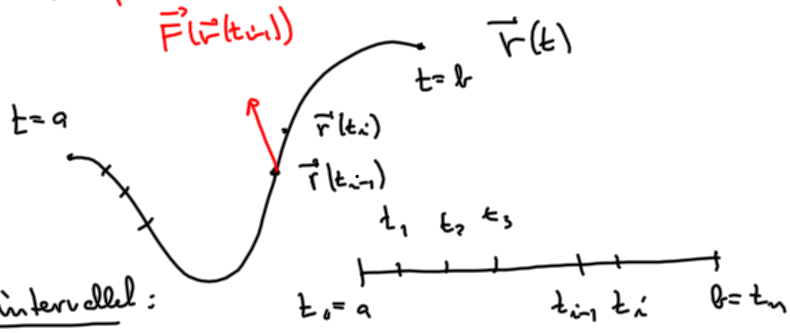


$A = F\cdot\cos v\cdot s$

$= \vec F\cdot\vec s$

Hva gjør vi når kraften ikke er konstant og veien ikke er rett?

Deler opp i småbiter der vi kan regne kraften som konstant og veien som rett.



Arbeidet A_i over det i-te intervallet:

$$A_i \approx \underbrace{\vec F(\vec r(t_{i-1}))}_{k_i}\cdot\underbrace{\vec v(t_{i-1})(t_i-t_{i-1})}_{s_i}$$

Totalt arbeid:

$$A = \sum \underbrace{\vec F(\vec r(t_{i-1}))\cdot\vec v(t_{i-1})}(t_i-t_{i-1}) \longrightarrow \int_a^b \vec F(\vec r(t))\cdot\vec v(t)\,dt$$

Riemannsum til
$\vec F(\vec r(t))\cdot\vec v(t)$

Definisjon: Anta at $\mathcal{C}$ er en stykkevis glatt kurve i $\mathbb{R}^n$ parametrisert av $\vec{r}:[a,b]\to\mathbb{R}^n$, og la $\vec{F}: A\to\mathbb{R}^n$ være et kontinuerlig vektorfelt definert på et område $A\subset\mathbb{R}^n$ som inneholder $\mathcal{C}$. Da definerer vi linjeintegralet $\int_{\mathcal{C}}\vec{F}\cdot d\vec{r}$ ved

$$\int_{\mathcal{C}}\vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\underbrace{\vec{v}(t)}_{\vec{r}\,'(t)}\,dt$$

forutsatt at det riste integralet eksisterer.

Eksempel: $\vec{r}(t) = t\vec{i} + e^{-t}\vec{j} + t^2\vec{k}$, $t\in[0,1]$

$$\vec{F}(x,y,z) = xy\vec{i} + x\vec{j} + z^2\vec{k}$$

$$\vec{F}(\vec{r}(t)) = F(t, e^{-t}, t^2) = te^{-t}\vec{i} + t\vec{j} + t^4\vec{k}$$

$$\vec{v}(t) = \vec{r}\,'(t) = \vec{i} - e^{-t}\vec{j} + 2t\vec{k}$$

$$\int_{\mathcal{C}}\vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{v}(t)\,dt =$$

$$= \int_0^1 (te^{-t}\vec{i} + t\vec{j} + t^4\vec{k})\cdot(\vec{i} - e^{-t}\vec{j} + 2t\vec{k})\,dt$$

$$= \int_0^1 (\cancel{te^{-t}} - \cancel{te^{-t}} + 2t^5)\,dt$$

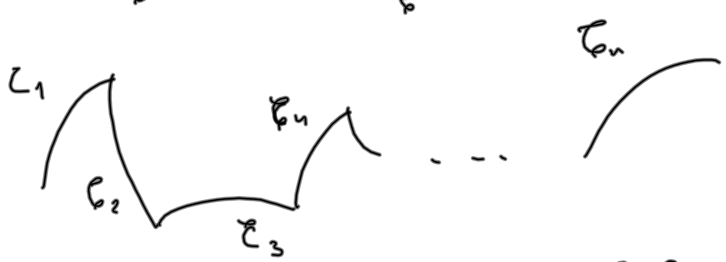
$$= \int_0^1 2t^5\,dt = \left[2\frac{t^6}{6}\right]_0^1 = \underline{\underline{\frac{1}{3}}}$$

Regneregler:

a) $\int_{\mathcal{C}}(\vec{F}+\vec{G})\cdot d\vec{r} = \int_{\mathcal{C}}\vec{F}\cdot d\vec{r} + \int_{\mathcal{C}}\vec{G}\cdot d\vec{r}$

b) $\int_{\mathcal{C}}(\vec{F}-\vec{G})\cdot d\vec{r} = \int_{\mathcal{C}}\vec{F}\cdot d\vec{r} - \int_{\mathcal{C}}\vec{G}\cdot d\vec{r}$

c) $\int_{\mathcal{C}}(c\vec{F})\cdot d\vec{r} = c\int_{\mathcal{C}}\vec{F}\cdot d\vec{r}$, for alle $c\in\mathbb{R}$

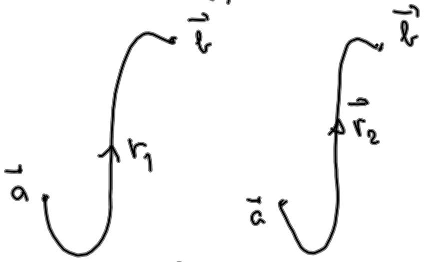


Hvis $\mathcal{C}$ kan deles opp i delkurver $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$, så er

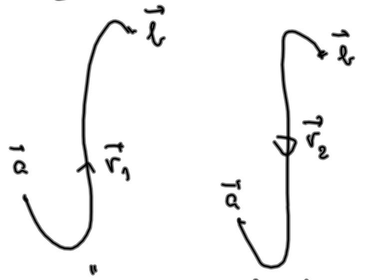
$$\int_{\mathcal{C}}\vec{F}\cdot d\vec{r} = \int_{\mathcal{C}_1}\vec{F}\cdot d\vec{r} + \int_{\mathcal{C}_2}\vec{F}\cdot d\vec{r} + \ldots + \int_{\mathcal{C}_n}\vec{F}\cdot d\vec{r}$$

Anta at $\vec r_1$ og $\vec r_2$ er to parametriseringer av den samme kurven $\mathcal{C}$. Hva med

$$\int_{\mathcal{C}_1} \vec F\cdot d\vec r \quad \text{og} \quad \int_{\mathcal{C}_2} \vec F\cdot d\vec r \ ?$$



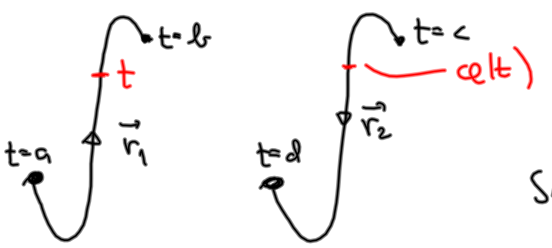
Hvis gjennomløpningsretningen er den samme, så $\int_{\mathcal{C}_1} \vec F\cdot d\vec r = \int_{\mathcal{C}_2} \vec F\cdot d\vec r$



Hvis parametriseringene har motsatt retning, så

$$\int_{\mathcal{C}_1} \vec F\cdot d\vec r = -\int_{\mathcal{C}_2} \vec F\cdot d\vec r$$

"Bevis" for motsatt retning:



$\vec r_1 : [a,b] \to \mathbb{R}^n$

$\vec r_2 : [c,d] \to \mathbb{R}$

Ser at $\varphi(b) = c$

$\varphi(a) = d$

Siden $\vec r_1(t) = \vec r_2(\varphi(t))$

$\vec V_1(t) = \vec V_2(\varphi(t))\,\varphi'(t)$

$$\int_{\mathcal{C}_1} \vec F\cdot d\vec r_1 = \int_a^b \vec F(\vec r_1(t))\cdot \vec v_1(t)\,dt = \int_a^b \vec F(\vec r_2(\varphi(t)))\cdot \vec v_2(\varphi(t))\,\varphi'(t)\,dt$$

$s = \varphi(t)$

$ds = \varphi'(t)\,dt$

$\varphi(a) = d$

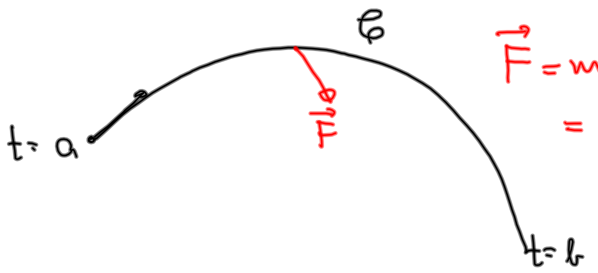
$\varphi(b) = c$

$$= \int_d^c \vec F(\vec r_2(s))\cdot \vec v_2(s)\,ds = -\int_c^d \vec F(\vec r_2(s))\cdot \vec v_2(s)\,ds$$

$$= -\int_{\mathcal{C}_2} \vec F\cdot d\vec r_2$$

"Physics for glupies II":
gloopies?

Newtons annen lov:



$$\vec F = m\vec a = m\vec r\,''$$

Kinetisk energi: $E_k = \frac{1}{2}mv^2$

Sammenheng: $\underbrace{\int_{\mathcal{C}} \vec F \cdot d\vec r}_{\text{tilført arbeid}} = \underbrace{\frac{1}{2}mv(b)^2 - \frac{1}{2}mv(a)^2}_{\text{økning i kinetisk energi}}$.

Bevis: $\displaystyle\int_{\mathcal{C}} \vec F \cdot d\vec r = \int_a^b (m\vec a(t))\cdot \vec v(t)\,dt = \frac{m}{2}\int_a^b \underbrace{2\vec a(t)\cdot \vec v(t)}\,dt$

Mellomregning: $v(t)^2 = \vec v(t)\cdot \vec v(t)$

Derivere: $(v(t)^2)' = \vec a(t)\cdot \vec v(t) + \vec v(t)\cdot \vec a(t) = \overbrace{2\vec a(t)\cdot \vec v(t)}$

Altså:
$$\int_{\mathcal{C}} \vec F\cdot d\vec r = \frac{m}{2}\int_a^b 2\vec a(t)\cdot\vec v(t)\,dt = \frac{m}{2}\int_a^b (v(t)^2)'\,dt$$
$$= \frac{m}{2}\Big[v(t)^2\Big]_{t=a}^{t=b} = \frac{m}{2}v(b)^2 - \frac{m}{2}v(a)^2$$
$$= \frac{1}{2}mv(b)^2 - \frac{1}{2}mv(a)^2$$

Gradienter og potensialer (3.5)

$\varphi : \mathbb{R}^n \to \mathbb{R}$

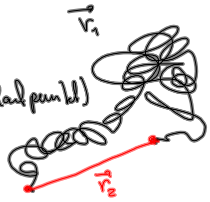
$$\nabla\varphi(\vec{x}) = \left(\frac{\partial\varphi}{\partial x_1}(\vec{x}), \frac{\partial\varphi}{\partial x_2}(\vec{x}), \ldots, \frac{\partial\varphi}{\partial x_n}(\vec{x})\right)$$

Hva blir linjeintegralet $\int_{\mathcal{C}} \nabla\varphi \cdot d\vec{r}$? $\quad \vec{r} : [a,b] \to \mathbb{R}^n$

$$\int_{\mathcal{C}} \nabla\varphi \cdot d\vec{r} = \int_a^b \underbrace{\nabla\varphi(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{(\varphi(\vec{r}(t)))'}\, dt = \int_a^b (\varphi(\vec{r}(t)))'\, dt$$

$$= \left[\varphi(\vec{r}(t))\right]_a^b = \varphi(\vec{r}(b)) - \varphi(\vec{r}(a))$$

$$= \varphi(\text{sluttpunkt}) - \varphi(\text{startpunkt})$$



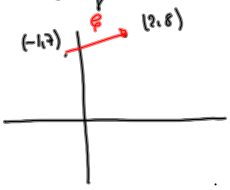
Setning: Dersom φ er en deriverbar funksjon og $\vec{r} : [a,b] \to \mathbb{R}^n$ er en stykkevis glatt kurve, så er

$$\int_{\mathcal{C}} \nabla\varphi \cdot d\vec{r} = \varphi(\vec{r}(b)) - \varphi(\vec{r}(a))$$

$$\int_{\mathcal{C}_1} \nabla\varphi \cdot d\vec{r} = \int_{\mathcal{C}_2} \nabla\varphi \cdot d\vec{r}$$

Integralet er altså uavhengig av veien; to kurver som starter i samme punkt gir samme verdi for integralet.

Eksempel: Vi skal integrere $\vec{F}(x,y) = 2xy^2\vec{i} + 2x^2y\vec{j}$ langs den rette linjen fra $(-1,7)$ til $(2,8)$.



Hvis $\varphi = x^2y^2$, så er

$$\nabla\varphi = 2xy^2\vec{i} + 2x^2y\vec{j} = \vec{F}(x,y)$$

Dermed:

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} \nabla\varphi \cdot d\vec{r} = \varphi(2,8) - \varphi(-1,7)$$

$$= 2^2 \cdot 8^2 - (-1)^2 \cdot 7^2 = \text{et interessant tall.}$$

Spørsmål: Når er $\vec{F}$ en gradient?

Lemma: Dersom $\vec{F} = (F_1, F_2, \ldots, F_n)$ er en gradient, så må

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

for alle i,j i alle punkter der $\vec{F}$ er definert.

Bevis: Hvis $\vec{F}$ er en gradient, så er $\vec{F} = \nabla\varphi$ i.e.

$$(F_1, F_2, \ldots, F_n) = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \ldots, \frac{\partial\varphi}{\partial x_n}\right)$$

Dermed:

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial x_i}\right) = \frac{\partial^2\varphi}{\partial x_j \partial x_i}$$

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial\varphi}{\partial x_j}\right) = \frac{\partial^2\varphi}{\partial x_i \partial x_j}$$

OBS: Det er ikke nødvendigvis slik at hvis $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ i hele definisjonsområdet til $\vec{F}$, så er $\vec{F}$ en gradient (det kommer an på hvordan definisjonsområdet ser ut).

Eksempel: $\vec{F}(x,y) = \overbrace{(3x^2y - ye^{-x} + 1)}^{F_1}\vec{i} + \overbrace{(x^3 + e^{-x} + 2y)}^{F_2}\vec{j}$

Er det mulig at dette feltet er en gradient?

Må sjekke om $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

$$\underbrace{\frac{\partial F_1}{\partial y} = 3x^2 - e^{-x} \qquad \frac{\partial F_2}{\partial x} = 3x^2 - e^{-x}}_{\text{Ja, mulig gradient}}$$