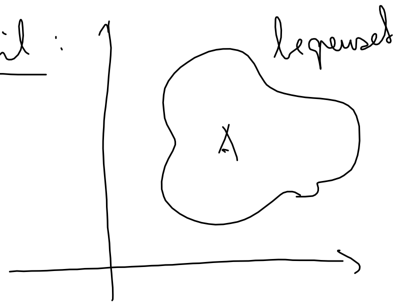
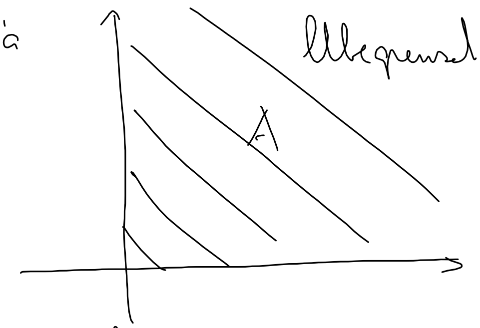
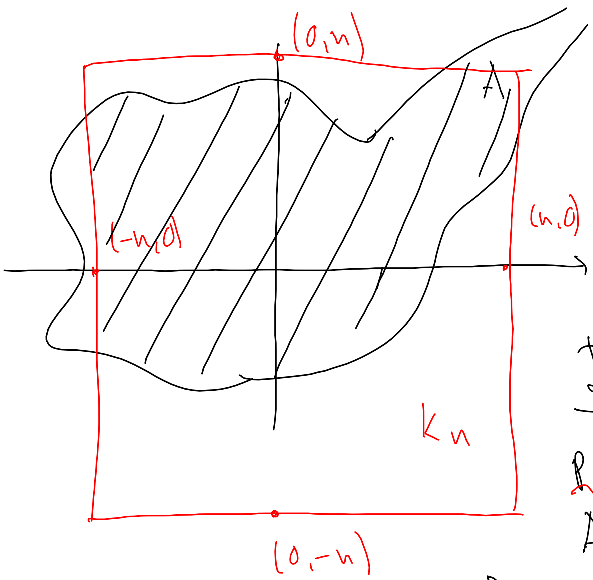
Uegentlige integraler (6.8)

Hittil: 

Nå 



Anta $f \geq 0$

$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap K_n} f(x,y)\,dx\,dy$$

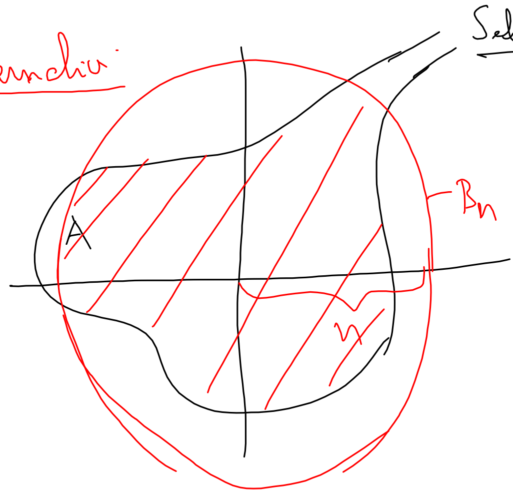
forutsatt at grensen finnes.

Definisjon: Anta f er en positiv, kontinuerlig funksjon og $A \cap K_n$ er Jordan-målbar for alle n. Da definerer vi

$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap K_n} f(x,y)\,dx\,dy$$

forutsatt at grenseverdien finnes. I så fall sier vi at integralet konvergerer (hvis ikke sier vi at det divergerer)

Alternativ:



Setning:

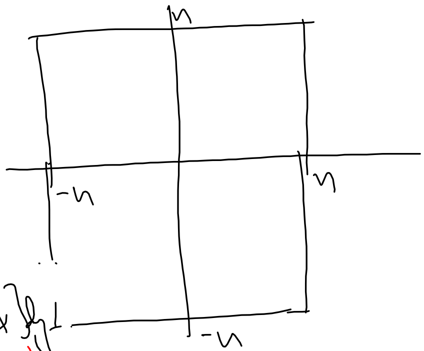
$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap B_n} f(x,y)\,dx\,dy$$

Eksempel: Regn ut $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. Husk: $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \lim_{n\to\infty} \underbrace{\int_{-n}^{n} e^{-x^2}\,dx}_{I_n} = \lim_{n\to\infty} I_n$.

Regner ut

$$J = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$$

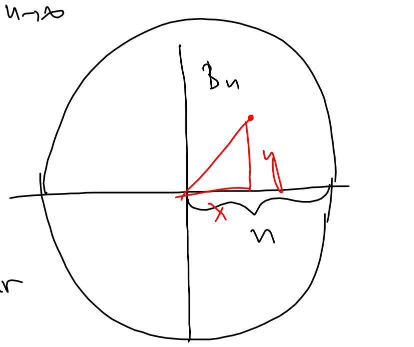
på to måter.



$$\text{I: } J = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \lim_{n\to\infty} \iint_{K_n} e^{-x^2-y^2}\,dx\,dy$$

$$= \lim_{n\to\infty} \int_{-n}^{n}\left[\int_{-n}^{n} \underbrace{e^{-x^2-y^2}}_{e^{-x^2}\cdot e^{-y^2}}\,dx\right]dy = \lim_{n\to\infty} \int_{-n}^{n} e^{-y^2}\underbrace{\left[\int_{-n}^{n} e^{-x^2}\,dx\right]}_{I_n}dy$$

$$= \lim_{n\to\infty} \int_{-n}^{n} e^{-y^2}\cdot I_n\,dy = \lim_{n\to\infty} I_n \underbrace{\int_{-n}^{n} e^{-y^2}\,dy}_{I_n} = \lim_{n\to\infty} I_n^2 = \underline{I^2}$$



$$\text{II: } J = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \lim_{n\to\infty} \iint_{B_n} e^{-x^2-y^2}\,dx\,dy$$

$$= \lim_{n\to\infty} \int_0^n\left[\int_0^{2\pi} e^{-r^2}\cdot r\,d\theta\right]dr = \lim_{n\to\infty} 2\pi \int_0^n e^{-r^2}\cdot r\,dr$$

$$= \lim_{n\to\infty} 2\pi \int_0^{n^2} e^{-u}\tfrac{1}{2}\,du = \lim_{n\to\infty} \pi \left[-e^{-u}\right]_0^{n^2}$$

$$= \pi \lim_{n\to\infty} \left[\underbrace{-e^{-n^2}}_{\to 0} + 1\right] = \underline{\underline{\pi}}$$

$e^{-x^2-y^2} = e^{-(x^2+y^2)} = e^{-r^2}$

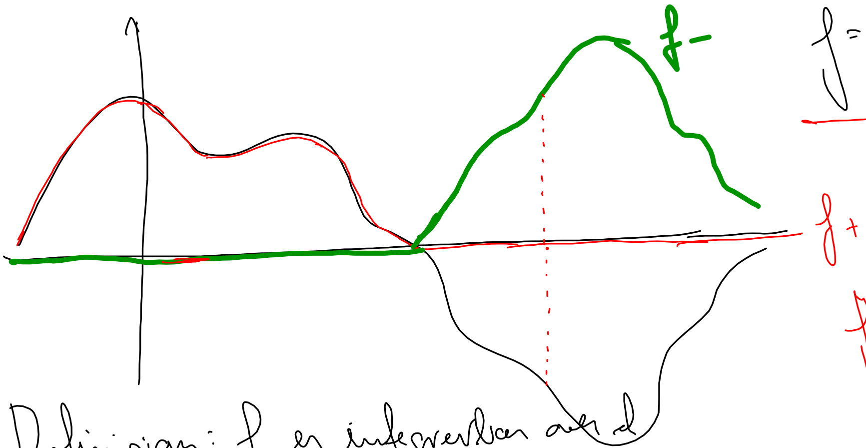
$u = r^2$, $du = 2r\,dr$

$r\,dr = \frac{1}{2}du$

Sammenligner svarene: $I^2 = \pi \Rightarrow I = \sqrt{\pi}$

Altså: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \underline{\sqrt{\pi}}$

Hva hvis betingelsen $f \geq 0$ ikke er oppfylt?



$f = f_+ - f_-$ der $f_+, f_- \geq 0$

$$f_+(x,y) = \begin{cases} f(x,y) & \text{når } f(x,y) \geq 0 \\ 0 & \text{ellers} \end{cases}$$

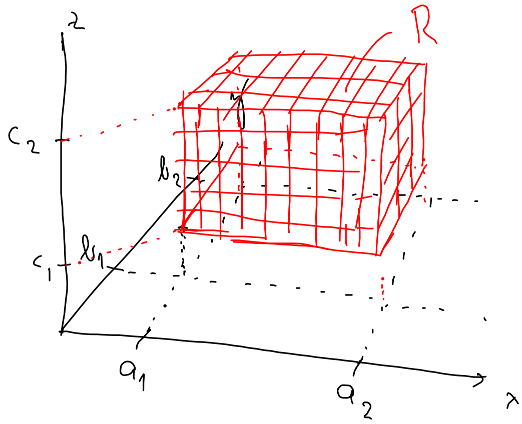
$$f_-(x,y) = \begin{cases} 0 & \text{når } f(x,y) \geq 0 \\ -f(x,y) & \text{når } f(x,y) \leq 0 \end{cases}$$

Definisjon: f er integrerbar over et område A dersom både f_+ og f_- er integrerbare over A, dvs. at begge integralene $\iint_A f_+ \,dx\,dy$ og $\iint_A f_- \,dx\,dy$ konvergerer. I så fall definerer

$$\iint_A f(x,y)\,dx\,dy = \iint_A f_+(x,y)\,dx\,dy - \iint_A f_-(x,y)\,dx\,dy$$

Trippelintegraler (16.9)

$$\iiint_A f(x,y,z)\,dx\,dy\,dz \quad , \quad A \subseteq \mathbb{R}^3$$



$R = \{(x,y,z) : a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2\}$

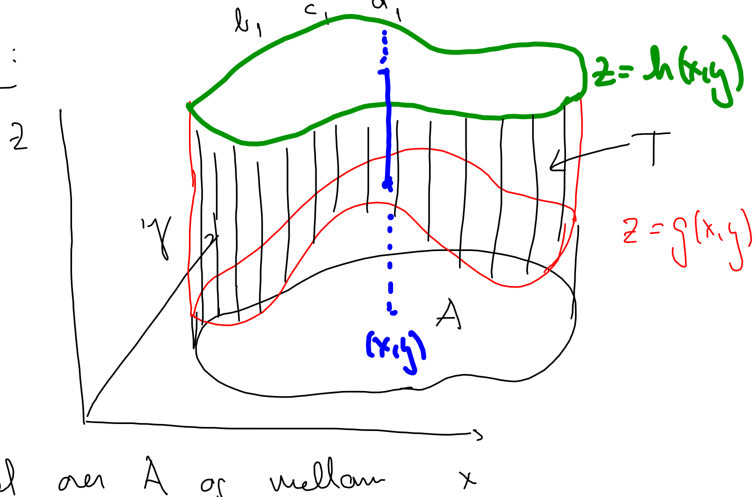
$\iiint_R f(x,y,z)\,dx\,dy\,dz =$ grensen av øvre og nedre trappesummer når oppdelingen blir finere og finere

Itereret integrasjon:

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{a_1}^{a_2}\left[\int_{b_1}^{b_2}\left[\int_{c_1}^{c_2} f(x,y,z)\,dz\right]dy\right]dx$$

$$= \int_{b_1}^{b_2}\left[\int_{c_1}^{c_2}\left[\int_{a_1}^{a_2} f(x,y,z)\,dx\right]dz\right]dy \overset{\text{osv.}}{=}$$

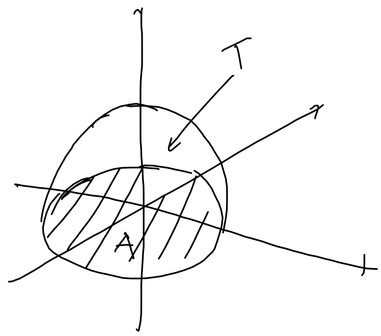
Mer avansert:



T er området over A og mellom funksjonsgrafene $z = h(x,y)$, $z = g(x,y)$.

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iint_A \left[\int_{g(x,y)}^{h(x,y)} f(x,y)\,dz\right] dx\,dy$$

Eksempel: Regn ut $\iiint_T z\,dx\,dy\,dz$ der T er området over xy-planet og inni kulen med radius 1 om origo



Nedre flate: $z = 0$

Øvre flate: halvkulen: $x^2+y^2+z^2 = 1$

$$z^2 = 1 - x^2 - y^2 \Rightarrow z = \sqrt{1-x^2-y^2}$$

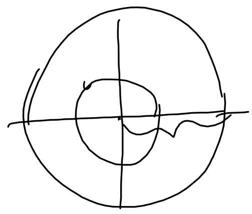
$$\iiint_T z\,dx\,dy\,dz = \iint_A \left[\int_0^{\sqrt{1-x^2-y^2}} z\,dz\right] dx\,dy$$

$$= \iint_A \left[\frac{1}{2}z^2\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} dx\,dy = \frac{1}{2}\iint_A (1-x^2-y^2)\,dx\,dy$$

$(1-(x^2+y^2))$ $(1-r^2)$

bytter til polarkoordinater

$$\frac{1}{2}\int_0^1\left[\int_0^{2\pi} (1-r^2)r\,d\theta\right] dr =$$



$$= \frac{1}{2}\cdot 2\pi\int_0^1 (1-r^2)r\,dr = \pi\int_0^1 (r-r^3)\,dr$$

$$= \pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{4}$$

Volum: $\mathrm{Vol}(T) = \iiint_T 1\,dx\,dy\,dz$

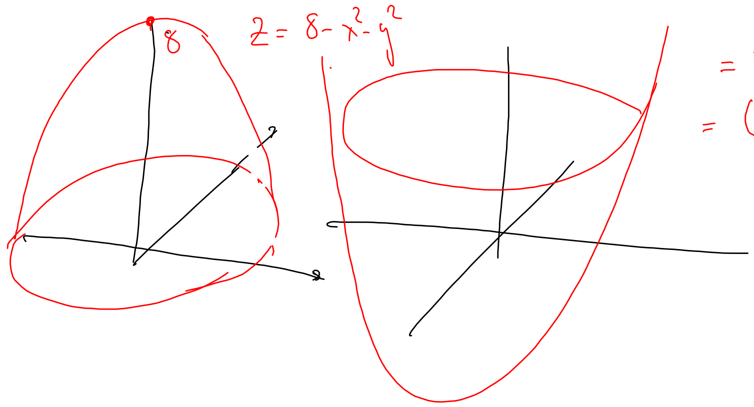
Eksempel: Finn volumet avgrenset av flatene:

$$z = 8 - x^2 - y^2 = 8-(x^2+y^2)$$

$$z = x^2 + 4x + y^2 - 8y$$

$$= x^2+4x+4+y^2-8y+16-20$$

$$= (x+2)^2 + (y-4)^2 - 20$$



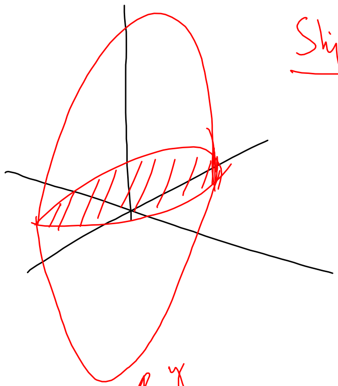
Skjæringskurve: $8 - x^2 - y^2 = x^2 + 4x + y^2 - 8y$

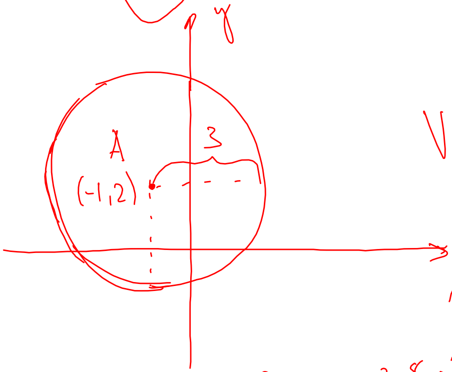
$$8 = 2x^2 + 4x + 2y^2 - 8y \quad | : 2$$

$$4 = x^2 + 2x + y^2 - 4y$$

$$\underbrace{x^2+2x+1}_{(x+1)^2} + \underbrace{y^2-4y+4}_{(y-2)^2} = 4+1+4$$

$$(x+1)^2 + (y-2)^2 = 3^2 \quad \text{Sirkel}$$





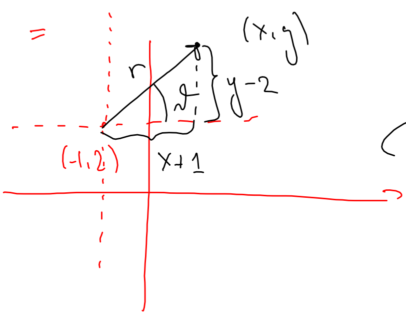
$$V = \iiint_T 1\,dx\,dy\,dz$$

$$= \iint_A \left[\int_{x^2+4x+y^2-8y}^{8-x^2-y^2} 1\,dz \right] dx\,dy$$

$$= \iint_A \left[z \right]_{z=x^2+4x+y^2-8y}^{z=8-x^2-y^2} dx\,dy$$

$$= \iint_A \left[8 - x^2 - y^2 - (x^2+4x+y^2-8y) \right] dx\,dy = \iint_A \left[8 - 2x^2 - 2y^2 - 4x + 8y \right] dx\,dy$$

=



$x+1 = r\cos\theta$ $\qquad x = -1 + r\cos\theta$

$y-2 = r\sin\theta$ $\qquad y = 2 + r\sin\theta$

$$= \int_0^3 \int_0^{2\pi} \left[r \quad \theta \quad \right] r\,d\theta\,dr$$