# Konservative felt

**Vektorfelt:** $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$, linjeintegraler: $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$

**Gradient:** $\vec{F} = \nabla \varphi$, $\int \nabla\varphi \cdot d\vec{r} = \varphi(\vec{r}(b)) - \varphi(\vec{r}(a))$

**Teorem:** Anta at $\vec{F}$ har kontinuerlige partiellderiverte.

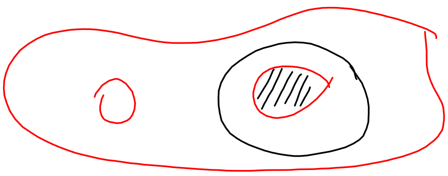
(i) Hvis $\vec{F}$ er en gradient i $A$, så er

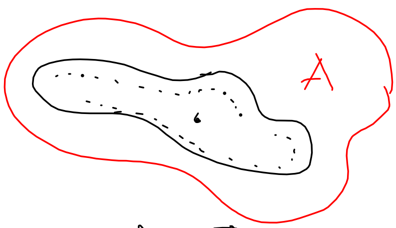
$$\frac{\partial F_i}{\partial x_j}(\vec{x}) = \frac{\partial F_j}{\partial x_i}(\vec{x}) \quad \text{for alle } \vec{x} \in A \text{ og alle } i,j$$

(ii) Dersom $A$ er enkeltsammenhengende og

$$\frac{\partial F_i}{\partial x_j}(\vec{x}) = \frac{\partial F_j}{\partial x_i}(\vec{x}) \quad \text{for alle } \vec{x} \in A \text{ og alle } i,j$$

så er $\vec{F}$ en gradient.






**Bevis:** a) Anta at $\vec{F}$ er en gradient, $\vec{F} = \nabla\varphi$.

Da er $F_i = \frac{\partial \varphi}{\partial x_i}$, $F_j = \frac{\partial \varphi}{\partial x_j}$. Dermed er

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial \varphi}{\partial x_i}\right) = \frac{\partial^2 \varphi}{\partial x_j \partial x_i} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial x_j}\right) = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

Kinetisk energi: $\frac{1}{2} m v^2 = E_k$

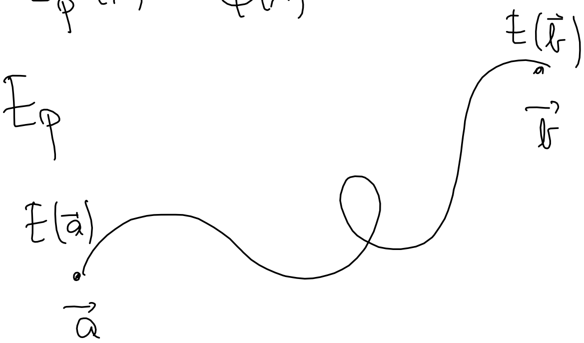
Potensiell energi: $\vec{F} = \nabla \varphi$, $E_p(\vec{x}) = -\varphi(\vec{x})$

Total energi: $E = E_k + E_p$

$$\int \vec{F} \cdot d\vec{r} = \frac{1}{2} m v^2(b) - \frac{1}{2} m v^2(a)$$

$$\|$$

$$\int \nabla \varphi \cdot d\vec{r} = \varphi(\vec{r}(b)) - \varphi(\vec{r}(a))$$

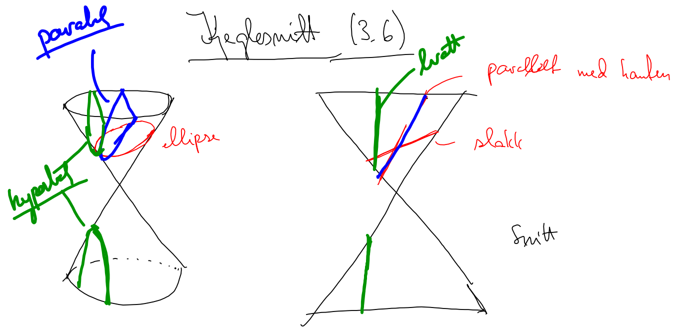



$$\frac{1}{2} m v^2(b) - \frac{1}{2} m v^2(a) = \varphi(\vec{r}(b)) - \varphi(\vec{r}(a))$$

$$\underbrace{-\varphi(\vec{r}(b)) + \frac{1}{2} m v^2(b)}_{E_t(b)} = \underbrace{-\varphi(\vec{r}(a)) + \frac{1}{2} m v^2(a)}_{E_t(a)}$$

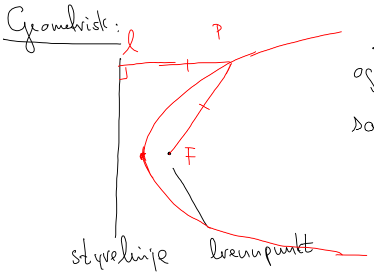
Samme totale energi før og etter (og underveis).

# Kjeglesnitt (3.6)



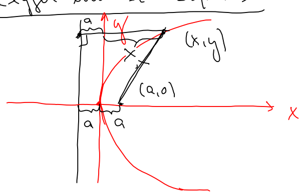

## Parabler

Geometrisk:




Parabelen med brennpunkt F og styrelinje l er kurven som består av alle de punktene som har samme avstand til F som til linjen l.

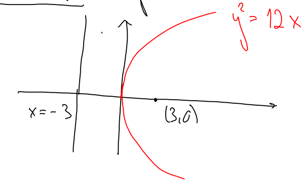
Legger inn et aksekors:

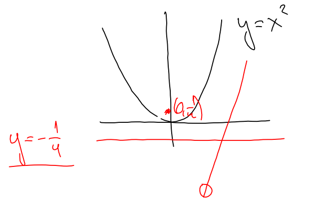



$$\sqrt{(x-a)^2+(y-0)^2} = x+a$$
$$\Downarrow$$
$$(x-a)^2+y^2 = (x+a)^2$$
$$\Downarrow$$
$$\cancel{x^2}-2xa+\cancel{a^2}+y^2 = \cancel{x^2}+2xa+\cancel{a^2}$$
$$\boxed{y^2 = 4ax}$$

Ligningen for en parabel med brennpunkt $(a,0)$ og styrelinje $x=-a$ er $y^2=4ax$. Vi kaller $a$ brennvidden til parabelen.

Eksempel: $y^2 = 12x = 4\cdot 3x$ : Parabel med brennvidde 3.

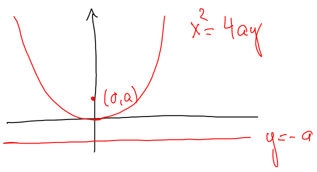


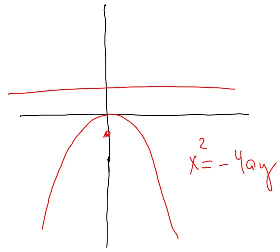



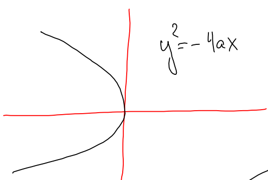

$x^2 = y = 4\cdot\frac{1}{4}\cdot y$

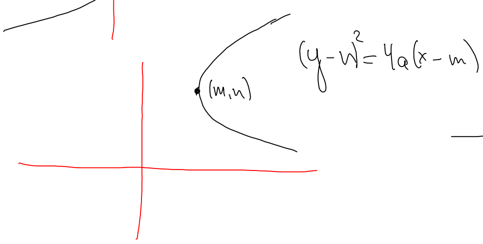
$a=\frac{1}{4}$

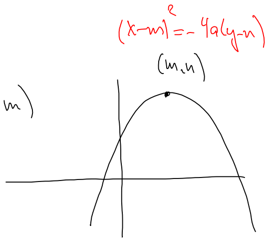
## Parabler med andre innretninger:



Eksempel: Hva slags kurve fremstiller ligningen:

$$y^2 - 4y - 8x + 8 = 0$$

Fullfører kvadratet:

$$\underbrace{(y^2 - 4y + 4)}_{(y-2)^2} - 4 - 8x + 8 = 0$$

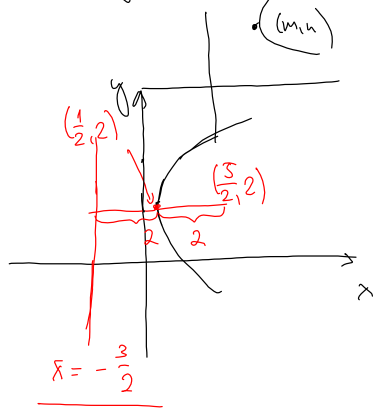
$$(y-2)^2 = 8x - 4$$

$$(y-2)^2 = 8\left(x - \frac{1}{2}\right)$$

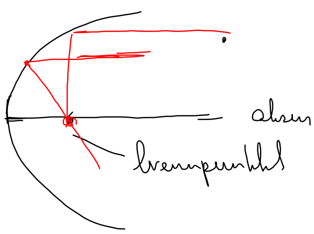
$$(y-2)^2 = 4 \cdot \underset{\uparrow\, a}{2}\left(x - \frac{1}{2}\right)$$

$$(y-n)^2 = 4a(x-m)$$

$(m,n)$



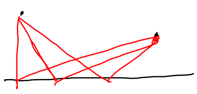

Refleksjonsegenskap



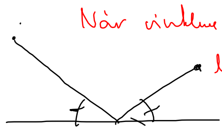

lysstråler som kommer inn parallelt med aksen reflekteres gjennom brennpunktet.

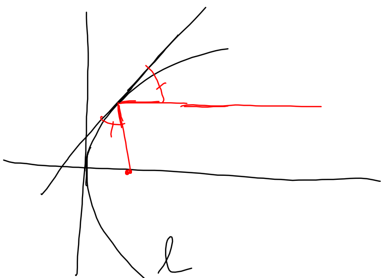
Fysikk: Lysstrålens bane er et lokalt minimum.

Matematikk.

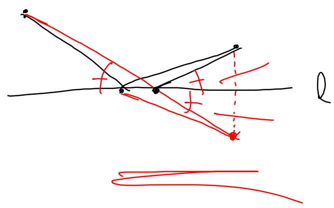


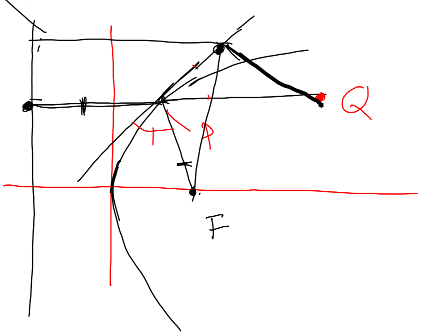
Korteste vei via linjen? Når vinklene er like store!





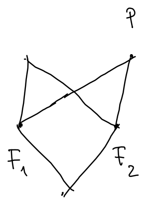

Smart knep for å finne korteste vei




P er det punktet på tangenten som gjør gangveien fra Q til P kortest mulig, og dermed er det denne veien lyset følger.

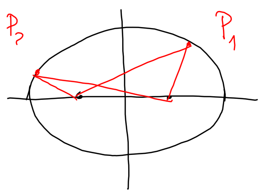
# Ellipser



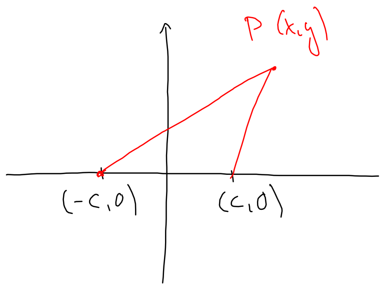

$$|PF_1| + |PF_2| = 2a$$

lengde $a$

**Definisjon:** Ellipsen med brennpunkt i $F_1$ og $F_2$ og med store halvakse $a$ er samlingen av alle punkter $P$ slik at




$$|PF_1| + |PF_2| = 2a$$




$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Downarrow$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

$$\Downarrow$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$$\cancel{x^2} + 2xc + \cancel{c^2} + \cancel{y^2} = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + \cancel{x^2} - 2xc + \cancel{c^2} + \cancel{y^2}$$

$$\cancel{4}xc = \cancel{4}a^2 - \cancel{4}a\sqrt{(x-c)^2+y^2}$$

$$a\sqrt{(x-c)^2+y^2} = a^2 - xc$$

$$\underbrace{a^2\left[(x-c)^2 + y^2\right] = a^4 - 2a^2xc + x^2c^2}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad b = \sqrt{a^2 - c^2}$$